Housing and Commuting:

The Theory of Urban Residential Structure

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APPENDIX 1.2.B
BASIC URBAN MODEL EXERCISES

Part I. Bid Functions and Density Functions

I-1. The slope of the bid-price function, $P\{u\}$, is $-t/H$. Use Equation (35) to derive general expression for the slope of the bid-rent function, $R\{u\}$.

I-2. Derive expressions for the slopes of the bid-rent function, $R\{u\}$, and the population density function, $D\{u\}$, in the case of Cobb-Douglas utility and housing production functions.

I-3. Equation (48) provides an expression for the population function, $N\{u\}$, with Cobb-Douglas utility and housing production functions. Derive and explain the slope of this function.

I-4. Derive an expression for the second derivative of the bid-price function, $P\{u\}$, with a Cobb-Douglas utility function.

I-5. Derive expressions for the second derivatives of the bid-rent function, $R\{u\}$, and the population density function, $D\{u\}$, in the case of Cobb-Douglas utility and housing production functions.

Part II. Comparative Statics

II-1. Prove that the closed-model comparative static derivative $dR\{u\} / d\overline{R}$, which is given by Equation (64), is positive.

II-2. Derive an expression for the comparative static derivative $d\overline{u} / dY$ in a basic closed urban model. Prove that this derivative is positive. [Hint: One approach is to use Equations (40) and (55) to show that if $d\overline{u} / dY$ is negative, $dD\{u\} / dY$ also must be negative, which is a contradiction; if both density and the size of the urban area increase, population cannot remain constant.]

II-3. Prove that with Cobb-Douglas utility and housing production functions,

(a) in an open urban model, $dR\{u\} / dt < 0$;

(b) in a closed urban model, $dR\{u\} / dt < 0$ for $u < u^*$ and $dR\{u\} / dt < 0$ for $u > u^*$ where $0 < u^* < \overline{u}$. [Hint: Use the population integral.]

II-4. Derive and evaluate the signs of expressions for the following comparative static derivatives in a basic open urban model with Cobb-Douglas utility and housing production functions:

$d\overline{u} / dY, d\overline{u} / dt, d\overline{u} / dU^*, d\overline{u} / d\overline{R}, dR\{u\} / dY, dR\{u\} / d\overline{R}, dR\{u\} / dt,$ and $dN\{u\} / dY$.

II-5. Derive and evaluate the signs of expressions for the following comparative static derivatives in a basic closed urban model with Cobb-Douglas utility and housing production functions: $d\bar{u} / dY$, $d\bar{u} / d\bar{R}$, and $dR\{u\} / d\bar{R}$.

II-6. Suppose that, over time, the commuting speeds in Drivealot decline because the popularity of non-work trips goes up, even during rush hour. Use comparative statics analysis (and the simplifying assumptions in question II-7.) to derive the impact of this response on the physical size of Drivealot and on the height of Drivealot's bid function, $P\{u\}$.

II-7. Empirical studies of land rents find a similar pattern in many cities over time, namely, declining land rents at the city center and increasing land rents in the suburbs. Some people argue that the main cause of this pattern is the fact that incomes are increasing over time. Evaluate this argument using comparative statics analysis from a basic urban model. You may answer this question using graphs and intuition if you like, but you should be as precise as possible. You may also use a highly simplified urban model (with one household type, Cobb-Douglas utility functions, and so on)

Part III. Bid Functions with Alternative Utility Functions

III-1. Derive bid-price and bid-rent functions for a basic urban model with a utility function of the following form:

$$U = a_1 \ \ln \{Z - \beta H + \omega\} + a_2 \ \text{In} \ \{H\}$$

where the $a's, \beta,$ and ω are constants. Express the bids either as functions of U^*, the fixed utility level, or $\bar{u}$, the outer edge of the urban area. Assume that the housing production function is Cobb-Douglas.

III-2. Solve an urban model with a CES utility function. Specifically,

(a) Derive a price-distance function when households have a CES utility function (and all other assumptions for a basic urban model hold):

$$U = \left(Z^{\delta} + H^{\delta}\right)^{1/\delta}$$

Assume that δ > 0 and remember that demand functions are not altered by a monotonic transformation of the utility function.

(b) Derive the corresponding rent-distance function (assuming Cobb-Douglas housing production).

III-3. Assume that leisure time, *S*, is an argument in a household's utility function, that the utility function is Cobb-Douglas, and that total time available, *T*, must equal work time, *W*, plus *S* plus time spent commuting, which is *u/M*, where *M* is commuting speed (miles per hour). In addition, assume that the round-trip operating costs of commuting equal t_0

per mile. Derive a bid-price function based on these assumptions. Compare it to the bid-price function derived in the text.

III-4. Suppose households have a Stone-Geary utility function:

$$U = \left(Z - S_Z\right)^{1-\alpha}\left(H - S_H\right)^{\alpha}$$

Prove that it is not possible to find an explicit form for the price-distance function with this utility function.

III-5. In an urban model, assumptions about the form of the household utility function (or about the form of the housing demand function) play a large role in determining the form of the $P\{u\}$ function. Derive this function using at least two different sets of assumption about household utility (or demand). Explain the differences between the two (or more) $P\{u\}$ functions you derive.

Part IV. Alternative Urban Models

IV-1. All workers in Bigtwo have identical Cobb-Douglas utility functions and identical skills. Bigtwo has only two employers, each of which is located in the CBD and each of which provides free buses to its employees. Firm A provides fancy buses with a smooth, quiet ride and with many forms of entertainment, such as movies, television, computers, and music. Firm B provides ordinary buses, which are not so smooth and quiet and which provide no entertainment at all. Workers are perfectly mobile, so they must obtain the same utility level, regardless of where they work. You have observed that employees of both firms live in neighborhoods located exactly u^* miles from the CBD.

(a) Which group of workers, employees of firm A or firm B, live outside u^* ?

(b) Which group of workers has higher wages? How much higher?

IV-2 Naderville has a greenbelt, that is, a wooded park, on a ridge that runs in a circle around the CBD. The distance between the CBD and the greenbelt is u^* miles. All residents of Naderville have the same incomes, utility functions, and commuting costs. Moreover, they are all very proud of the greenbelt and like to look at it. In fact, residents of Naderville are willing to pay for a better view of the greenbelt from their home. The quality of this view declines with distance from the greenbelt.

More specifically, each resident of Naderville has the following utility function:

$$U = Z^{\alpha_1} H^{\alpha_2} \left| u - u^* \right|^{-a_3}$$

where all the α's are positive and $a_1 + a_2 = 1$.

(a) Derive the price-distance function for Naderville, expressed as a function either of the fixed utility level, U^*, or of the outer edge of the urban area, $\bar{u}$.

(b) Derive the condition under which the price-distance function in Naderville will be positively sloped for locations inside u^*.

IV-3. Write down a standard urban model with Cobb-Douglas utility and production functions and two income classes. Assume that these two classes have the same utility functions and that commuting costs consist of time costs, which are proportional to income, and on operating costs, which are the same for both groups.

(a) The bid functions of the two groups cross at distance *u from the CBD. Prove that the higher-income class lives outside u^*.

(b) Derive an expression for the population in each group.

(c) Suppose operating costs are also proportional to income. Does this affect your answer to part (a)? Explain.

IV-4. Derive the so-called land constant for an urban model with a street grid, with vertical and horizontal arteries through the CBD, and with two vertical arteries that do not go through the CBD.

IV-5. The country of Equalomia is dedicated to equality. The economy of Equalomia is based on free enterprise, subject to various rules designed to make market outcomes more equal. In recognition of the need for work incentives, the government does not insist on complete equality and households in Equalomia fall into one of two income classes, which both contain the same number of people. Households in the richer income class earn exactly twice as much as households in the poorer class. The government has been unwilling to diverge from complete equality in the housing market, however, and every household in Equalomia receives exactly kY^* units of housing services, where k is a constant set by the government and Y* is average household income. Thus, households can compete with each other for existing apartments, but all apartments contain kY^* units of housing services.

Your job is to describe urban structure in Equalomia. You have discovered that everyone works in the city center; that per-mile commuting costs are constant within an income class; that households have Cobb-Douglas utility functions; that housing services are produced with a fixed coefficients production function using land and capital (that is, $H_s\{u\} = aL\{u\} = bK\{u\}$, where $H_s\{u\}$ is total housing services supplied at distance u and a and b are constants); and that the amount of residential land is proportional to distance (that is, $L\{u\} = \phi u$, where ϕ is a constant).

IV-6. A bid function, $P\{u\}$ can be derived when amenities, $a\{u\}$, are included in the utility function. Two derivations appear in the literature. The first derivation comes from Polinsky and Shavell. They write an indirect utility function:

$$V^* = V\{P\{u\}, Y - T\{u\}, a\{u\}\},$$

where V^* is a fixed utility level (to reflect household mobility),

$V_1 = \partial V / \partial P < 0$, $V_2 = \partial V / \partial(Y - T\{u\}) > 0$, and $a\{u\}$ is defined so that $V_3 = \partial V / \partial P > 0$.

Differentiating this function with respect to *u*, recognizing that $\partial V^* / \partial P = 0$, and solving for $\partial P / \partial u = P'\{u\}$ yields:

$$P'\{u\} = \frac{V_2}{V_1} T'\{u\} - \frac{V_3}{V_1} a'\{u\}$$

An alternative derivation begins with the following direct utility function:

$$U^* = U\{Z, H, a\{u\}\}$$

The first-order conditions of this problem imply that

$$P'\{u\} = \frac{\frac{\partial U / \partial a}{\lambda} a'\{u\} - T'\{u\}}{H} = \frac{\frac{\partial U / \partial a}{\partial U / \partial Z} a'\{u\} - T'\{u\}}{H} = \frac{MB_a a'\{u\} - T'\{u\}}{H}$$

where λ is the Lagrangian multiplier and MB_a is the marginal benefit from a unit of a in dollar terms.

Prove that these two expressions for $P'\{u\}$ are equivalent.

IV-7. Use the final equation from question IV-6 to show that high-income households will live outside low-income households whenever $P'\{u\} < 0$ and

$$\frac{\left(\frac{\partial MB_a}{\partial Y} a'\{u\} - \frac{\partial T'\{u\}}{\partial Y}\right) Y}{MB_a a'\{u\} - T'\{u\}} < \frac{\partial H}{\partial Y} \frac{Y}{H} .$$

Interpret this condition.

IV-8. New Orleans has an unusual amenity, namely, elevation. This amenity, labeled *e*, is measured in distance above or below sea level. Elevation is an amenity in New Orleans because of the probability of flooding. New Orleans is protected by levies and pumps. If the levies around the city are breached, the lowest places will flood first, followed by the higher places if the breach and resulting flood are severe enough, particularly if the water overwhelms the pumps. The people of New Orleans don't seem too concerned about this; so far as they are concerned, floods occur so seldom that they are not worth worrying about. Based on past experience, however, home insurance companies care a lot, and they charge more to provide full insurance, including flood coverage, for houses in low-lying neighborhoods. Since flood insurance is required by the government (for the purposes of this problem!), people care about their elevation because it affects the cost of their insurance. To be more specific, housing, *H*, is measured in units of (quality-adjusted) square feet, and the price per unit of *H* is *P*. The price of a house is $V = PH/r$, where *r* is the appropriate discount rate. The price of a home insurance policy is set as a percentage of *V*. This percentage, *I*, is a function of *e*. Because the probability of flooding declines with *e*, so does the price of insurance, so that $I'\{e\} = dI / de < 0$.

Your job is to derive an expression for the price per unit of *H*, labeled *P*, as a function of *u* (= distance from the CBD, which is the only worksite) and *e*. Use the information in the previous paragraph to specify the household budget constraint. Assume that there is a single household type with a Cobb-Douglas utility function and that all households are homeowners. Make any other simplifying assumptions you want that are appropriate for a basic urban model. Derive and interpret expressions for $P\{u,e\}$ and for $\partial P/\partial e$.

IV-9. Ringcity is part of a large urban area with a beltway. Planners in the region are concerned that the pull of employment to this beltway will result in a "greybelt," defined as a section of the city where there is no residential or commercial development.

Your job is to determine the conditions under which a greybelt will arise, that is, under which there will be a ring in which the bid for land to be used for housing is below the bid for land in non-urban uses.

You may use the standard assumptions for an open urban model with Cobb-Douglas utility and housing production functions. Assume that the beltway is located u^* miles away from the CBD, that it circles the city at this distance, and that the employment along the beltway is constant. Assume that all commuting occurs on radial streets, that there is only one worker per household, that all workers (and households) are identical, and that workers can move between CBD jobs and jobs along the beltway. If you need to make any other assumptions to complete your derivation, just state them as clearly as possible.

Now derive a condition that must be satisfied for a greybelt to arise.

IV-10. Consider an urban model with a suburban employment ring. Suppose all workers are alike and mobile, so that they all achieve the same level of utility, regardless of whether they work in the CBD or in the suburban ring. Now suppose wages are twice as high in the CBD as in the ring. Assume that there is no “greybelt” in the middle of the urban area; in other words, the entire space between the two worksites is residential. Otherwise, you may make any reasonable simplifying assumptions (Cobb-Douglas utility functions, for example).

(a) Derive an expression for the boundary separating the residential zones of CBD workers and suburban workers.

(b) Add households with two workers, one in each worksite, and derive the boundaries of their residential area.