Envelopes for Economists: Housing Hedonics and Other Applications

An e-Book Edited by John Yinger

With contributions by: Il Hwan Chung, William D. Duncombe, Yue Hu,
Phuong Nguyen-Hoang, and Pengju Zhang

Chapter 1: Introduction to Envelope Mathematics and Its Applications in Economics

John Yinger
April 2020

1.0 Introduction

An envelope is a well-known mathematical concept with applications in chemistry, engineering, physics, graphic design, and, of course, economics.[1] This chapter presents the mathematics of simple envelopes and shows how this concept can be applied to one pioneering economic model from the nineteenth century. This chapter also develops some techniques for the derivation of envelopes in economic applications that will prove useful in later chapters.

The economic model in this chapter was developed by Johann Heinrich von Thünen in the early 1800s. This model introduced the concepts of bidding and sorting, which are at the heart of Parts 2 through 5 of this book. Although von Thünen never explicitly mentioned the concept of an envelope, this concept is implicit in the logic of his model. Moreover, the concept of an envelope is incorporated into the models that are his intellectual descendants, such as the bid-rent models that first appear in Alonso (1964) and the widely cited hedonic markets model developed by Rosen (1974).[2]

1.1 Deriving Envelopes

1.1.1. Introduction to the Mathematics of One-Parameter Envelopes

An envelope is a curve that is tangent to every member of an underlying family of curves. In the simplest case, this family can be defined by an implicit function F that has one parameter, α, and two variables, X and Y.[3] Each member of the "family" is associated with a different value for α. An envelope for this family, if it exists, is a function that meets the following two conditions:

$$F\{\alpha, X, Y\} = 0 \tag{1.1}$$

$$\frac{\partial F\{\alpha, X, Y\}}{\partial \alpha} = 0 \tag{1.2}$$

Consider the function[4]

$$F\{\alpha, X, Y\} = (1-\alpha)X + \alpha Y - \alpha(1-\alpha) = 0 \tag{1.3}$$

which has an X intercept at $X = \alpha$ and a Y intercept at $Y = (1 - \alpha)$. This function defines a family of straight lines with different values for the α parameter. To find the envelope, we can differentiate Equation (1.3) with respect to α and set the result equal to zero:

$$\frac{\partial F}{\partial \alpha} = -X + Y - 1 + 2\alpha = 0 \tag{1.4}$$

Solving equation (1.4) for α yields

$$\alpha = \frac{1 + X - Y}{2} \tag{1.5}$$

The final steps are to substitute Equation (1.5) into Equation (1.4) and to simplify the result, yielding the envelope:

$$(-X + Y - 1)^2 - 4X = 0 \tag{1.6}$$

An example of this function with “family members” based on values of α between 0.1 and 0.9 and of the resulting envelope is presented in Figure 1.1.

Not all one-parameter functions have envelopes, but many of them do. Another random example is

$$F\{\beta, x, y\} = y - \left(\ln\{\beta\}x^2 - \ln\{x\} - x^3 - \beta\right) = 0 \tag{1.7}$$

The derivative of Equation (1.7) with respect to β is

$$\frac{\partial F}{\partial \beta} = -\frac{x^2}{\beta} + 1 = 0 \tag{1.8}$$

The resulting envelope, which is pictured in Figure 1.2, is

$$y = \ln\{x^2\}x^2 - \ln\{x\} - x^2 - x^3 \tag{1.9}$$

1.1.2. The Ladder Problem

One entertaining and useful envelope derivation is the so-called ladder problem. Imagine a ladder leaning against a building. Now suppose the ladder slips, hopefully with no one on it. Then the ladder traces out an envelope of the highest points at each distance from the building. This problem is illustrated in Figure 1.3. The length of the ladder is L. The distance from the building is x_0. It follows that the y-intercept is $\sqrt{L^2 - {x_0}^2}$ and that the slope of the line is minus one multiplied by this intercept and divided by x_0. The resulting line to describe the ladder is

$$y = \sqrt{L^2 - {x_0}^2} - \left(\frac{\sqrt{L^2 - {x_0}^2}}{x_0}\right)x \tag{1.10}$$

We can now find the envelope of this equation using the approach in Equations (1.1) and (1.2), that is, by solving[5]

$$F\{x_0, x, y\} = y + \left(\frac{\sqrt{L^2 - x_0^{\ 2}}}{x_0} \right) (x - x_0) = 0 \tag{1.11}$$

and

$$\frac{\partial F}{\partial x_0} = \frac{x_0^{\ 3} - L^2 x}{x_0^{\ 2}\sqrt{L^2 - x_0^{\ 2}}} = 0 \tag{1.12}$$

Equation (1.12) indicates that $x_0^3 = L^2x$ or $x = x_0^3/(L^2)$. Substituting this expression into Equation (1.11) reveals that $y = [(L^2 – x_0^2)^{3/2}]/(L^2)$. We can now eliminate x^0 by raising each of these two expressions to the 2/3 power and adding them together:

$$\begin{aligned} y^{2/3} &= \frac{L^2 - x_0^{\ 2}}{L^{4/3}} \\ x^{2/3} &= \frac{x_0^{\ 2}}{L^{4/3}} \end{aligned} \tag{1.13}$$

The sum of these two terms, which is the desired envelope is:

$$y^{2/3} + x^{2/3} = L^{2/3} \tag{1.14}$$

An example of this envelope, with the ladders at various stages of slipping, is provided in Figure 1.4. This envelope describes one-quarter of a figure known as an “astroid,” leaving us one vowel short of a model from outer space.[6]

1.2 von Thünen

As indicated at the beginning of this chapter, it is entirely fitting to begin an investigation into the use of envelopes in economics by considering the work of Johann Heinrich von Thünen, who introduced a spatial dimension into economic analysis and invented the concepts of bidding and sorting.[7] Although von Thünen (1875) only considered an agricultural economy, the work of Alonso (1964) and others extended von Thünen’s analysis to an urban area, thereby creating the field of urban economics.[8] The literature that first defined this field forms the foundation of the analysis in Chapter 10 of this book. Moreover, as shown in Parts 2 and 3, bidding and sorting—and the associated envelopes—are crucial concepts for understanding the role of public services and amenities in a federal system, including the impact of these factors on housing prices and household location.

This section begins with an overview of von Thünen's contributions to economics, with a focus on his land-use model. The core of this model is a set of bid functions for land for various agricultural activities. The second part of the section turns to envelopes, and, in particular, shows how to derive envelopes for two modified versions of the von Thünen bid functions.

1.2.1 The von Thünen Model

Johann Heinrich von Thünen was born in northern Germany in 1783. He began his training in agriculture as a young man and then, "[b]eginning in 1803, he studied national economy in Goettingen."[9] In 1809, he purchased an estate in Mecklenburg-Schwerin, also in northern Germany, where he combined the practical management of his agricultural enterprises with his economic research. The first part of his most famous work was published in 1826. This work brought him immediate recognition as a scholar, and he received an honorary doctorate from the University of Rostock in 1830. He died in 1850. At his request, one of his equations was carved into his tombstone.[10]

Before proceeding, it is worth noting that von Thünen's contributions go well beyond his land-use theory. Indeed, a detailed review of his work by Samuelson (1983, p. 1468) begins by saying that von Thünen was an

> economist who met a payroll and, in recording and analyzing his Junker estate accounts, not only created *marginalism* and *managerial economics*, but also elaborated one of the first models of *general equilibrium* and did so in terms of realistic *econometric* parameters. (Emphasis in original)

These features of von Thünen's work are reviewed by Samuelson. The following discussion focuses on his land-use theory.[11]

Von Thünen's analysis explored the location of various agricultural activities around a market situated in an "isolated" city.[12] His key insight was that land rents reflect the returns from owning and employing land. His analysis of this insight was based on the assumptions (1) that this city was surrounded by a featureless plain upon which different activities could be located, (2) that agricultural products could be placed on wagons (or, in the case of livestock, herded) and taken straight to the market, and (3) that different activities had different returns per acre and different transportation costs.[13]

On the basis of these assumptions, therefore, von Thünen expressed land rent per acre per year, R, as

$$R = Yp - E - Yfk \tag{1.15}$$

where "E = production expenses per acre, including labor, supplies, and equipment; Y = yield in units of commodity per acre; p = market price per unit of commodity; f = freight rate, i.e., the cost of shipping a unit of commodity over the distance of one mile; and k = number of miles from the market" (Grotewold 1959, p. 349). In more modern terms, Equation (1.15) describes a "bid function," which is the amount a firm in a certain activity would pay for land at each distance from the market.[14]

The logic of these bid functions led von Thünen to recognize that each economic activity would locate in a ring around the central market. He argued that fresh milk and vegetable production would be in the inner-most ring, because these products have very high transportation costs, determined largely by the fact that they spoil quickly. (These two different commodities are located together because the manure from the milk cows keeps the land fertile for the vegetables.) In addition, “Since fresh milk and vegetables were required in the city, prices for them had to be so high that no other use of the land would have yielded a higher rent” (Grotewold 1959, pp. 349-350). In other words, the rents for these two activities are high near the market and fall off rapidly with distance from the market.

The second ring, according to von Thünen, would be for timber and fire wood productions, because these products were necessary but their weight led to high transportation costs. The following rings would be for grain and then livestock.[15] Overall, therefore, the products with lower values of (Yf) in Equation (1.7) locate in rings farther from the central market.

A stylized version of the von Thünen analysis is presented in Figure 1.5. This figure plots the relationship between land rents and distance from the central market, that is, the bid functions, for various activities identified by von Thünen, using numbers that are consistent with his assumptions about product prices and transportation costs. Note that the assumptions about (Yf) determine the steepness of the bid functions, and the assumptions about ($Yp - E$) determine their intercepts. The intercept and slope of the flattest bid-function determine the size of the cultivated area, with no economic activity beyond the point where this bid-function intersects the distance axis.[16]

This figure provides the essence of bidding and sorting, concepts discussed at great length in later chapters. Different types of activities bid different amounts for land (or housing) at different locations. The activity that bids the most at a given location wins the competition for land at that location. In other words, competition for land results in sorting, with different activities in different rings around the central market.

Moreover, this figure implicitly introduces the notion of an envelope in a land market, although von Thünen never explicitly referred to this concept, let alone this term.[17] The envelope in Figure 1.5 is the set of winning bids for land at every location, which are, of course, the only bids one actually observes. This envelope obviously becomes flatter as one moves farther from the market; with linear envelopes. To put it another way, activities with steeper bid functions will win the competition closer to the central market. In addition, note that this envelope is a series of line segments corresponding to the different types of farms, not a continuous function that is the solution to a version of Equation (1.1) and (1.2). As illustrated in the next section, a continuous envelope requires a continuous formulation of farm heterogeneity.

A final feature of Von Thünen's work that is an important part of his legacy is that he not only developed the main model discussed here, but he also explored several models with alternative assumptions. He considered the possibility, for example, that a river or some other geographical feature might interrupt his "featureless" plain, and he recognized that a smaller city might exist with its own trade area and its own pattern of sorting. These and other "modified conditions" are discussed in Grotewold (1959). Von Thünen's recognition of the role of smaller cities is particular noteworthy, because it foreshadows the development of urban models with more than one employment center—a topic explored in Chapter 10.[18]

1.2.2 Deriving a von Thünen Envelope, Method 1

Von Thünen's basic analysis can be extended to consider an even more heterogeneous set of activities, which makes it possible to derive the envelope of heterogeneous firms' bid functions. Figures 1.1 and 1.4 and the algebra behind them provide envelopes for families of downward-sloping linear curves. This section and the next two examine the question: Is it possible to derive an envelope for the linear von Thünen bid functions in Equation (1.15) and Figure 1.5.

As it turns out, answering this question does not improve upon the von Thünen model, which is based on detailed knowledge of the agricultural markets in which he participated.[19] Instead, this exercise sheds light on the mathematics of bid-function envelopes—and of the strong assumptions that are needed to generate linear bid functions. This section presents one simple model in the spirit of von Thünen's that leads to linear bid functions and shows one way to derive the associated bid-function envelope. The next section provides a similar model with an alternative envelope derivation.

First, assume that households have identical Cobb-Douglas utility functions with a positive utility parameter, α_i, for each of N goods, indexed with an i. Let Y be total income in the community per unit time. For convenience, let us assume that the unit of time for this and all other variables is one year. Then total demand in the community for good i is $\alpha_i Y/P_i$, where P_i is its price.[20] The α_is must add up to 1.0 to ensure that total spending on all N goods equals total income, Y.

Assume a uniform plain so that land for production u miles from a central market equals $2\pi u$.[21] Assume, like von Thünen, that goods are shipped directly to the central market at a cost of t_i per unit per mile.

Assume that production for good i per unit of land equals a production parameter θ_i divided by distance from the market, u. Assume as well, that each good is produced at only one location, that is, at one distance from the central market. Thus, total production for good i is $2\pi\theta_i$, regardless of the location of the farm. These assumptions describe a situation in which each farmer can only harvest a given amount of the crop. If more land is available, therefore, the farmer plants the crop less densely so that the amount he as to harvest stays constant.

Setting demand equal to supply, we have:

$$\frac{\alpha_i Y}{P_i} = 2\pi\theta_i \tag{1.16}$$

or

$$P_i = \frac{\alpha_i Y}{2\pi\theta_i} \tag{1.17}$$

Bids for land vary by product and location. A firm's bid equals its net return per unit of product at location *u* multiplied by the number of units produced at *u*. The net return per unit is price minus transportation cost. As a result, the bid for land, *R*, by a firm producing good *i* at location *u* is

$$\begin{aligned} R_i\{u\} &= \left(P_i - t_i u\right)(2\pi u)\left(\frac{\theta_i}{u}\right) = \left(2\pi\theta_i\right)\left(P_i - t_i u\right) \\ &= \left(2\pi\theta_i\right)\left(\frac{\alpha_i Y}{2\pi\theta_i} - t_i u\right) = \alpha_i Y - \left(2\pi\theta_i t_i\right)u \end{aligned} \tag{1.18}$$

This equation describes a linear bid function, such as the ones depicted in Figure 1.5. In fact, although the terminology is somewhat different, Equation (1.18) is equivalent to Equation (1.15)—except for the lack of production expenses, which are considered in the next section.

The final step in the analysis is to specify distributions for α_i, t_i, and θ_i. Because the product of t_i and θ_i appears in Equation (1.3), we can treat this product as a single parameter. Then we can find α_i by making use of the ladder envelope derived earlier. This approach does not have a specific economic interpretation, but it is perfectly consistent with both von Thünen's model and the standard approach to sorting, which is discussed at length in Chapter 4. A sorting equilibrium is one in which land goes to the highest bidder. As shown by Equation (1.18), the slope of a bid function is $-2\pi\theta_i t_i$. The firm with the steepest bid function, that is, with the highest value of $2\pi\theta_i t_i$, will win the competition for land right next to the central market. To keep the length of the "ladder" constant, the firm with the second-highest value of $2\pi\theta_i t_i$ will have a slightly lower bid for land right next to the market and will locate in the neighboring ring of land.

Equation (1.18) indicates that the height of a bid function right next to the market, is $\alpha_i Y$. As a result, a "slightly lower bid for land next to the market" corresponds to a slightly lower value for α_i. If α_i were constant across products, it would not be possible to derive an envelope. Instead, the bid function for the firm with the lowest value for $\theta_i t_i$ would have the same bid as every other firm right next to the central market—and a higher bid everywhere else. This observation provides a crucial piece of intuition for later chapters: An envelope cannot be derived unless the underlying bid functions (or functions for other economic concepts) actually cross.

Equation (1.18) also shows that firm i's bid per unit of land reaches a value of zero when $u = \alpha_i Y / 2\pi\theta_i t_i$. The distance between this x-intercept and the above y-intercept is:

$$D_i = \left(\frac{\alpha_i Y}{2\pi\theta_i t_i}\right)\sqrt{\left(2\pi\theta_i t_i\right)^2 + 1} \tag{1.19}$$

Now setting $D_i = D$ for all products and solving for α_i yields

$$a_i = \left(\frac{D 2\pi\theta_i t_i}{Y\sqrt{\left(2\pi\theta_i t_i\right)^2 + 1}}\right) \tag{1.20}$$

Equation (1.20) gives value of α_i that keeps the length of the bid-rent "ladder" constant for any given value of $\theta_i t_i$. Substituting this expression back into equation (1.18) yields the final ladder-consistent bid function:

$$R_i\{u\} = \alpha_i Y - \left(2\pi\theta_i t_i\right)u = \frac{D2\pi\theta_i t_i}{\sqrt{\left(2\pi\theta_i t_i\right)^2 + 1}} - \left(2\pi\theta_i t_i\right)u \tag{1.21}$$

This equation can be written in the same form as Equation (1.11). Recall that x_0 in that equation is the *x*-intercept, and *D* (length of bid function) in this application means the same thing as *L* (length of the ladder) in that case. As indicated above, the *x*-intercept in this model is $\alpha_i Y / 2\pi\theta_i t_i$ or, plugging in Equation (1.21),

$$x_0 = \left(\frac{D}{\sqrt{\left(2\pi\theta_i t_i\right)^2 + 1}}\right) \tag{1.22}$$

Substituting this expression into Equation (1.11) yields:

$$y = \sqrt{D^2 - \left(\frac{D}{\sqrt{\left(2\pi\theta_i t_i\right)^2 + 1}}\right)^2} - \left(\frac{\sqrt{D^2 - \left(\frac{D}{\sqrt{\left(2\pi\theta_i t_i\right)^2 + 1}}\right)^2}}{\left(\frac{D}{\sqrt{\left(2\pi\theta_i t_i\right)^2 + 1}}\right)}\right)x$$

$$= D\sqrt{1 - \frac{1}{\left(2\pi\theta_i t_i\right)^2 + 1}} - \left(\frac{\sqrt{1 - \frac{1}{\left(2\pi\theta_i t_i\right)^2 + 1}}}{\frac{1}{\sqrt{\left(2\pi\theta_i t_i\right)^2 + 1}}}\right)x \tag{1.23}$$

$$= \frac{D2\pi\theta_i t_i}{\sqrt{\left(2\pi\theta_i t_i\right)^2 + 1}} - \left(2\pi\theta_i t_i\right)x$$

which is obviously the same as Equation (1.18) using *x* instead of *u*. It follows directly, that Equation (1.14) provides the envelope for Equation (1.21). Using the notation from our von Thünen model, this envelope is

$$R\{u\} = \sqrt{D^2 - u^2} \tag{1.24}$$

This is the envelope illustrated in Figure 1.4.

1.2.3 Deriving a von Thünen Envelope, Method 2

As noted above, a family of curves cannot have an envelope unless the family members cross. In standard mathematical applications, parameters that differentiate the members of the family must generate this type of crossing on their own. In the example presented above, the assumption of constant bid-function "length" determines the value of the intercept for any given value of the slope parameters (and of Y). This link between intercept and slope is crucial; if all members of the family had the same intercept, then an envelope would not exist.

As we will see throughout this book, economic applications often generate a set of curves that differ in slope but have undefined intercepts. In cases like this, the envelope often cannot be derived without bringing in some economics to determine how the intercept changes as the slope changes. In a few cases, such as the derivation of the long-run average costs curves with specific production functions in Chapter 3, an envelope can be derived without reference to Equation (1.1) and (1.2). Most of the other applications in this book, however, require a method for deriving the intercept based on economic principles.

This approach can be applied to the analysis in the previous section by assuming that the utility parameter, α, is the same for every commodity and introducing the notion of production expenses, E, which, as discussed earlier, are included in the original Thünen model. More specifically, let us assume that production expenses per acre for a given product, E_i, are the same no matter where that product is produced. Moreover, let us simplify the notation by defining a single production parameter, $\gamma_i \equiv 2\pi\theta_i t_i$. With these assumptions we can re-write Equation (1.18) as

$$R_i\{u\} = \alpha Y - E_i - \gamma_i u \tag{1.25}$$

Now we can derive an envelope in two steps. The first step recognizes that if two bid functions cross at location u^*, then then an envelope exists only if the bid-function with the steeper slope has a larger intercept. More specifically, the intercept must change so that the height of the bid-function at u^* does not change when γ changes from γ_i to γ_{i+1}. Dropping the *i* subscript and switching to a continuous formulation, we can write this condition as:

$$\left.\frac{dP}{d\gamma}\right|_{u=u^*} = -dE - u(d\gamma) = 0 \tag{1.26}$$

or

$$\frac{dE}{d\gamma} = -u \tag{1.27}$$

This step is illustrated in Figure 1.6. Two bid functions with the form in Equation (1.25) cross ten miles from the central market. Bid Function 1 has a steeper slope, which corresponds to a larger value for γ. The intercept is $(\alpha Y - E)$, so Equation (1.27) indicates that a larger γ goes with a smaller E and hence with a larger intercept. Hence, the intercept for Bid Function 1 is larger than the intercept for Bid Function 2. Because αY does not change, the derivative of the intercept with respect to γ is just $-dE/d\gamma = u$. This derivative is highlighted in the figure.

Equation (1.27) shows how the intercept changes with γ at a given value of u, but it does not describe the entire envelope. A second step is needed. That's where some economics comes in.

The von Thünen model describes an equilibrium in which production activities with steeper bid functions locate closer to the market place. In other words, the equilibrium in this model is described by a monotonic, negative relationship between γ and u. We do not know the exact nature of the relationship, but we can approximate it. Consider the following linear approximation:

$$u = a + b\gamma;\ \ b < 0 \tag{1.28}$$

Substituting this expression into Equation (1.27) yields

$$\frac{dE}{d\gamma} = -(a + b\gamma) \tag{1.29}$$

The solution to this simple differential equation is

$$E = E_0 - a\gamma - \frac{b\gamma^2}{2} \tag{1.30}$$

where E_0 is a constant of integration. Now solving Equation (1.28) for γ and substituting the result and Equation (1.30) into Equation (1.25) gives us, after some simplification, the desired envelope:

$$R\{u\} = \alpha Y - E_0^* + \left(\frac{a}{b}\right)u - \left(\frac{1}{2b}\right)u^2 \tag{1.31}$$

where

$$E_0^* = E_0 + \frac{a^2}{b} - \frac{a^2}{2b} \tag{1.32}$$

This envelope and illustrative curves from the underlying bid functions are illustrated in Figure 1.7. This envelope is similar to the one in Figure 1.5, but reflects much more heterogeneity—with a specific formulation—than observed among the four types of farms in the earlier figure.

Figure 1.8 compares an envelope that arises using this approach with one that arises using the previous approach. These two envelopes are drawn with the same intercept and with parameters selected to make them as similar as possible. The two envelopes are similar, although the one based on the latter problem has somewhat greater curvature at small distances and somewhat lower curvature at large distances. Looking ahead, the main approach in this book is to derive as general a specification as possible for the envelope and then to estimate the parameters of the associated hedonic equilibrium, such as the expressions in parentheses in equation (1.31).

1.2.4 Deriving a von Thünen Envelope, Method 3

Another method for finding a bid-function envelope is given by Rouwendal (1984). This approach, which is discussed in more detail in Chapter 5, ensues that the family of bid functions has an envelope by specifying the constant term in Equation (1.25) or its equivalent as a function of the variable that differentiates one bid function from another. In Equation (1.25) this variable is γ. Rouwendal uses a quadratic specification for the constant term, so with his approach a bid-function for the Von Thünen model can be written (dropping the i subscript) as:

$$R\{u\} = \alpha_0 + \alpha_1\gamma + \alpha_2\gamma^2 - \gamma u \tag{1.33}$$

The derivative of Equation (1.33) with respect to γ is

$$\frac{dR}{d\gamma} = \alpha_1 + 2\alpha_2\gamma - u \tag{1.34}$$

Setting this derivative equal to zero leads to:

$$\gamma = \frac{u - \alpha_1}{2\alpha_2} \text{ or } u = \alpha_1 + 2\alpha_2\gamma \tag{1.35}$$

Substituting the expression for γ in Equation (1.35) into (1.33), yields the envelope of the bid functions, namely,

$$R\{u\} = \alpha^* + \left(\frac{\alpha_1}{2\alpha_2}\right)u - \left(\frac{1}{4\alpha_2}\right)u^2 \tag{1.36}$$

where α^* is a constant. This result is obviously equivalent to Equation (1.21), so it is illustrated by Figures 1.7 and 1.8.

Note that Equation (1.33) can be interpreted as a description of the hedonic equilibrium, that is, of the relationship between the factor that determines the relative bid function slope for a given agricultural activity, γ, and the location of that activity, u. In other words, the Rouwendal approach implicitly assumes that this equilibrium relationship is linear. This is, of course, the same assumption that was made by the previous approach. The difference is that the previous approach starts by assuming a form for the hedonic equilibrium and then derives the envelope, whereas the Rouwendal approach assumes a common form for the constant term in the underlying family of bid functions—an assumption made with no reference to theory—and then derives the implied form for the hedonic equilibrium. In any case, the resulting specification for the hedonic equation is the same in the two cases, so long as the assumed or implied form for the hedonic equilibrium is the same.

1.3 Conclusions

It is appropriate to begin a book on housing hedonics with a review of a model developed von Thünen (1875). Although this model applies to agricultural activities, not housing, it introduces the concepts of bidding and sorting and, implicitly draws on the concept of an envelope. Alonso (1964) applied von Thünen's concepts of bidding and sorting to the study of residential land in an urban area. By re-stating these concepts with continuous functions, Alonso also opened the door to a more formal treatment of bidding and sorting in a housing market with heterogeneous households and introduced the term "envelope" to describe the housing hedonic.

This chapter uses the von Thünen model to introduce the mathematical concept of an envelope as a tool for studying the allocation of heterogeneous activities across space. This is a heuristic exercise, not an attempt to improve on the von Thünen model. Indeed, the focus of the von Thünen model on discrete agricultural activities in discrete rings around a market is undoubtedly more realistic than a model in which this sorting is characterized using a continuous function of distance from the market. This chapter should therefore be interpreted as a source of information on various techniques for deriving envelopes using continuous mathematics.

As demonstrated by Figure 1.8, the techniques in this chapter do not all lead to the same envelopes. Moreover, the applications of these techniques in this chapter are relatively simple, and more general applications appear in subsequent chapters. For example, we will use these techniques to solve for the housing hedonic, that is, the envelope, when the hedonic equilibrium is not linear. This type of analysis appears in Chapters 5, 7, and 9 for housing hedonics based on public services and neighborhood amenities and in Chapter 10, for housing hedonics based on commuting costs.

References

Alonso, William. 1964. *Location and Land Use: Toward a General Theory of Land Rent*. Cambridge, MA: Harvard University Press.

Grotewold, Andreas. 1959. "Von Thünen in Retrospect." *Economic Geography* 35 (4) (October): 346-355.

Haggett. Peter. 1996. *The Geographer's Art*. Wiley-Blackwell.

Rosen, Sherwin. 1974. "Hedonic Prices and Implicit Markets: Product Differentiation in Pure Competition." *The Journal of Political Economy* 82 (1) (January/February): 34-55.

Rouwendal, Jan. 1992. "The Hedonic Price Function as an Envelope of Bid Functions: An Exercise in Applied Economic Theory." *Journal of Housing and the Built Environment* 7 (1): 59–80.

Samuelson, Paul A. 1983. "Thünen at Two Hundred." *Journal of Economic Literature* 21 (4) (December): 1468-1488.

White, Michele J. 1999. "Urban Areas with Decentralized Employment: Theory and Empirical Work." In *Handbook of Urban and Regional Economics, Volume 3, Applied Urban Economics*, edited by P. Cheshire and E. S. Mills (North-Holland): 1375-1412.

von Thünen, Johann Heinrich. 1875. *Der isolirte Staat in Beziehung auf Landwirhstschaft und Nationalökonomie*, 3rd ed. 4 vol., Berlin: Wiegandt, Hempel & Parey, 1875.

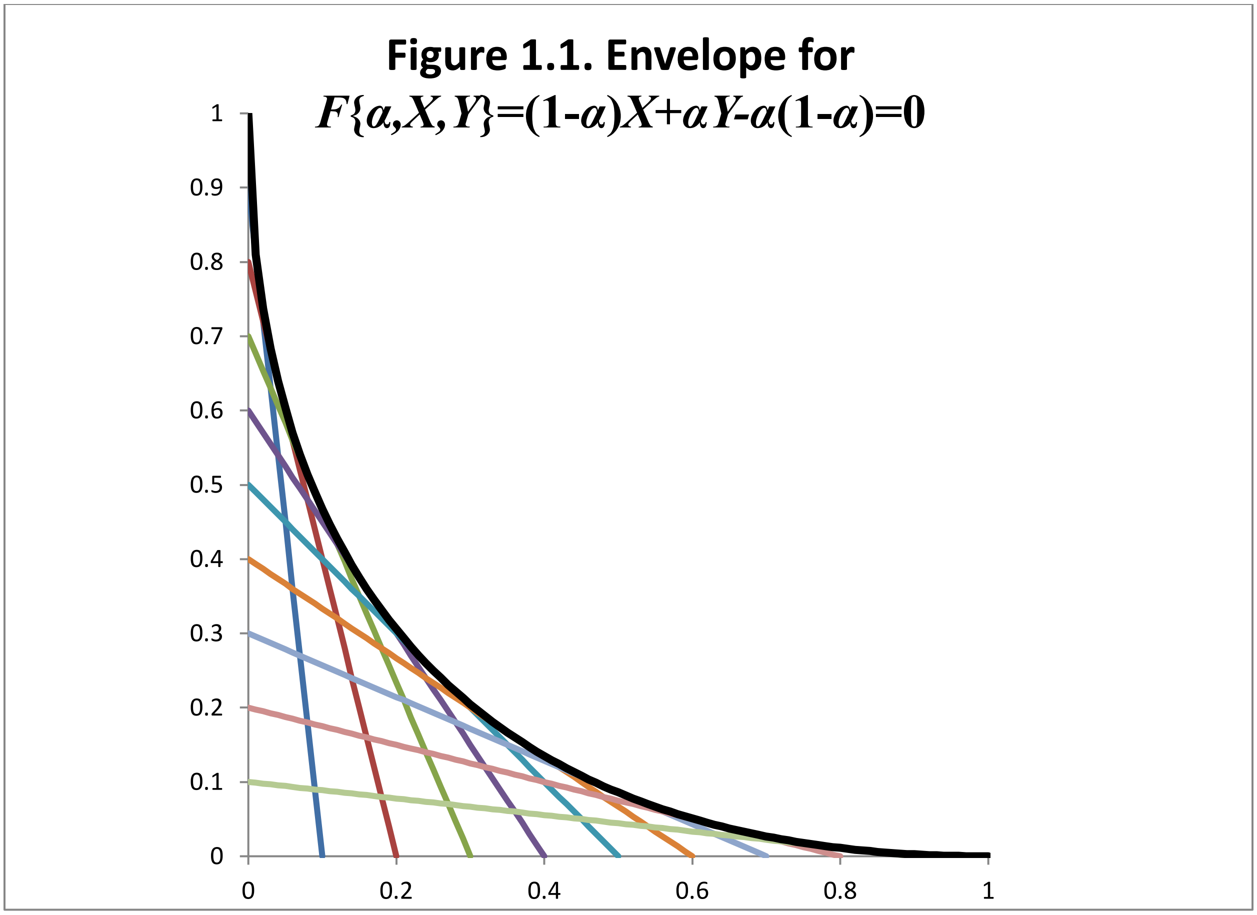


Figure 1.1. Envelope for
$F\{\alpha,X,Y\}=(1-\alpha)X+\alpha Y-\alpha(1-\alpha)=0$
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1

Figure 1.2. Envelope for $y = \ln\{\beta\}x^2 - \ln\{x\} - \beta - x^3$

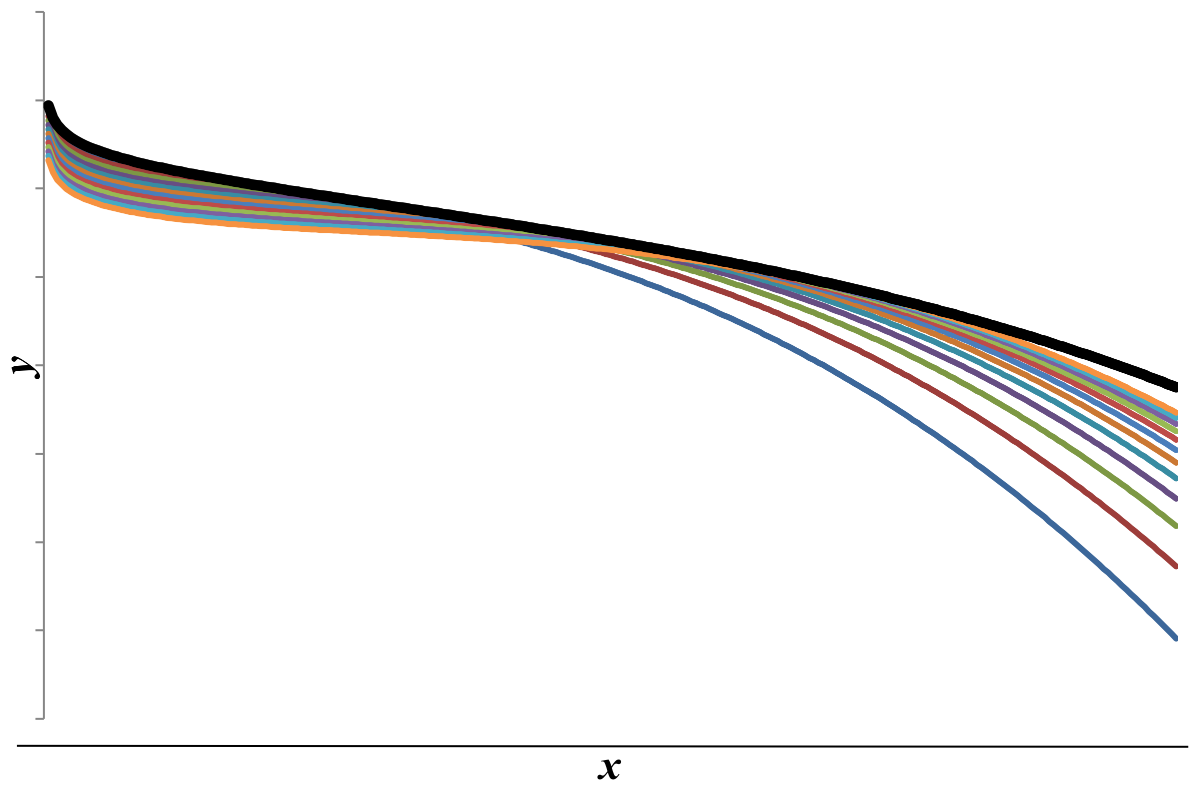
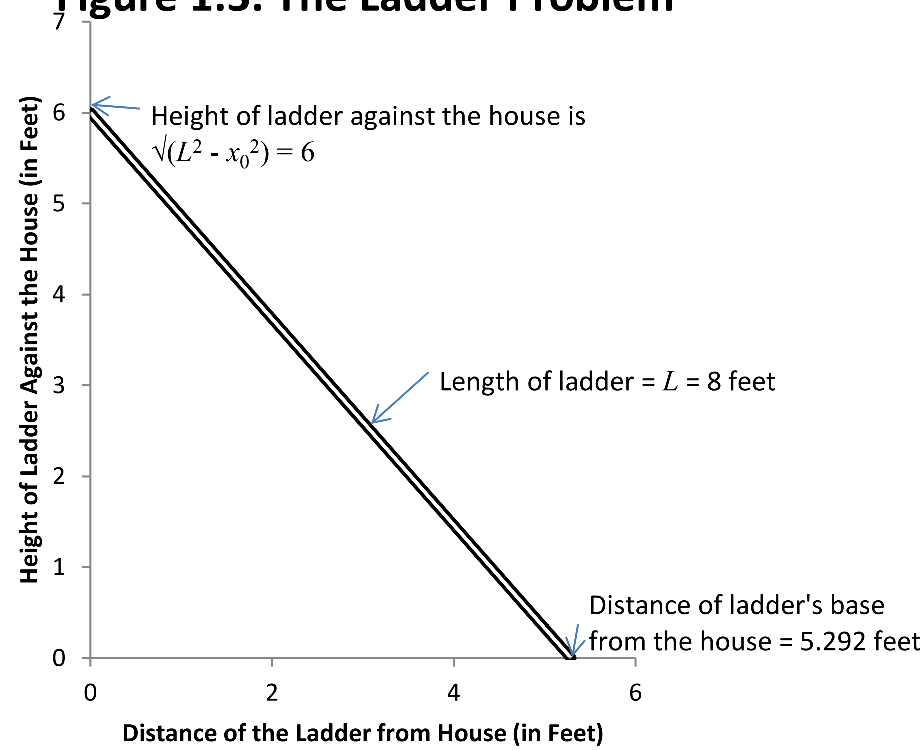


Figure 1.3. The Ladder Problem



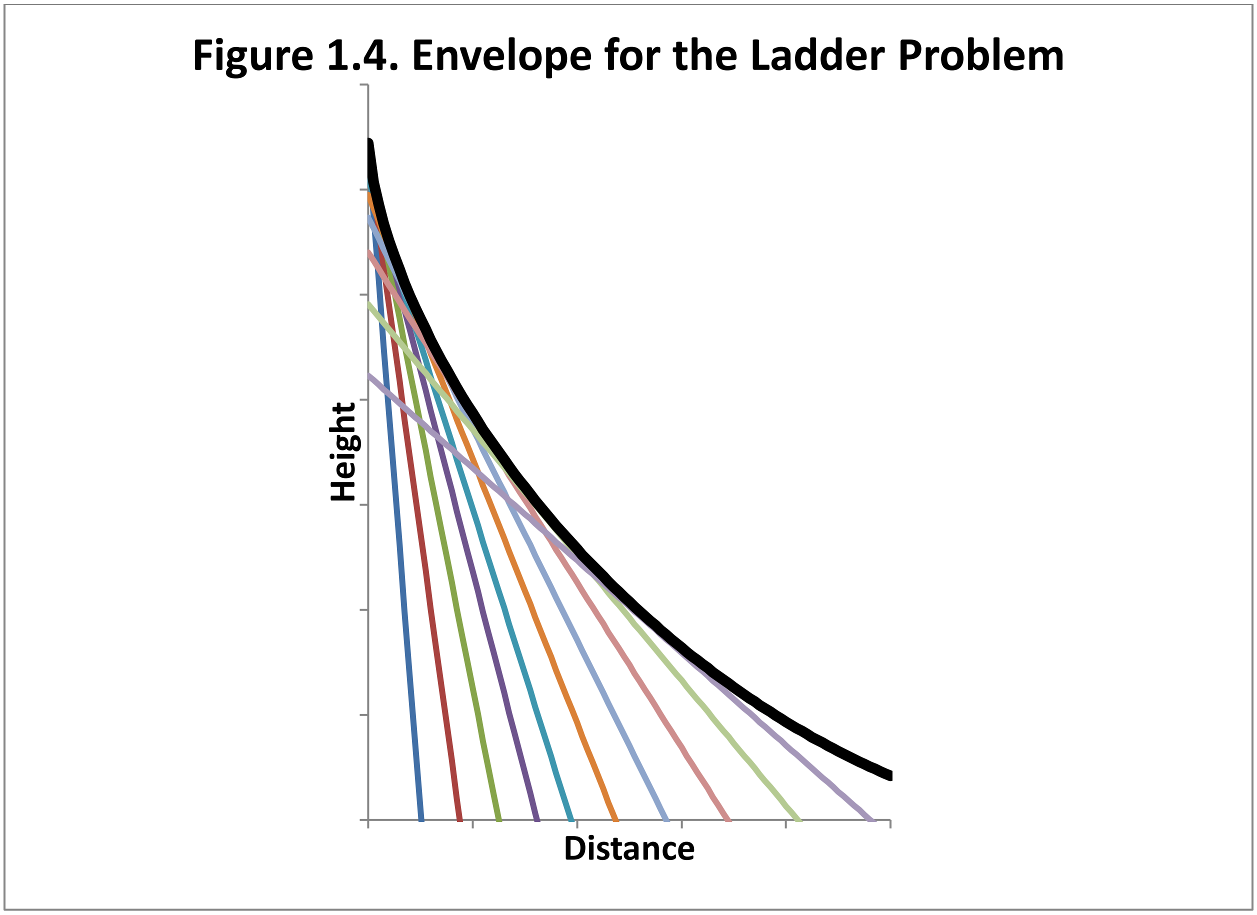


Figure 1.4. Envelope for the Ladder Problem
Height
Distance

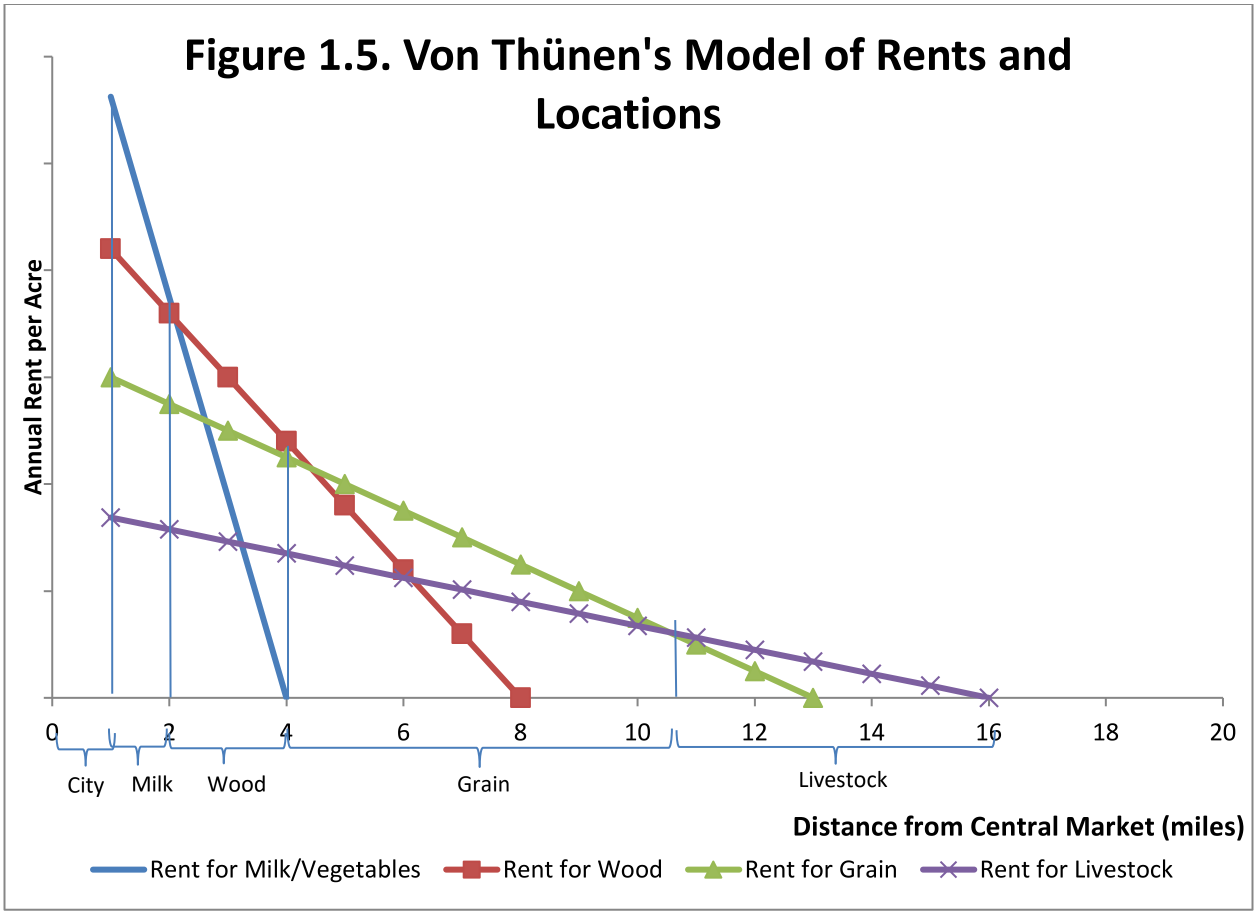


Figure 1.5. Von Thünen's Model of Rents and Locations
Annual Rent per Acre
0
2
4
6
8
10
12
14
16
18
20
City
Milk
Wood
Grain
Livestock
Distance from Central Market (miles)
Rent for Milk/Vegetables
Rent for Wood
Rent for Grain
Rent for Livestock

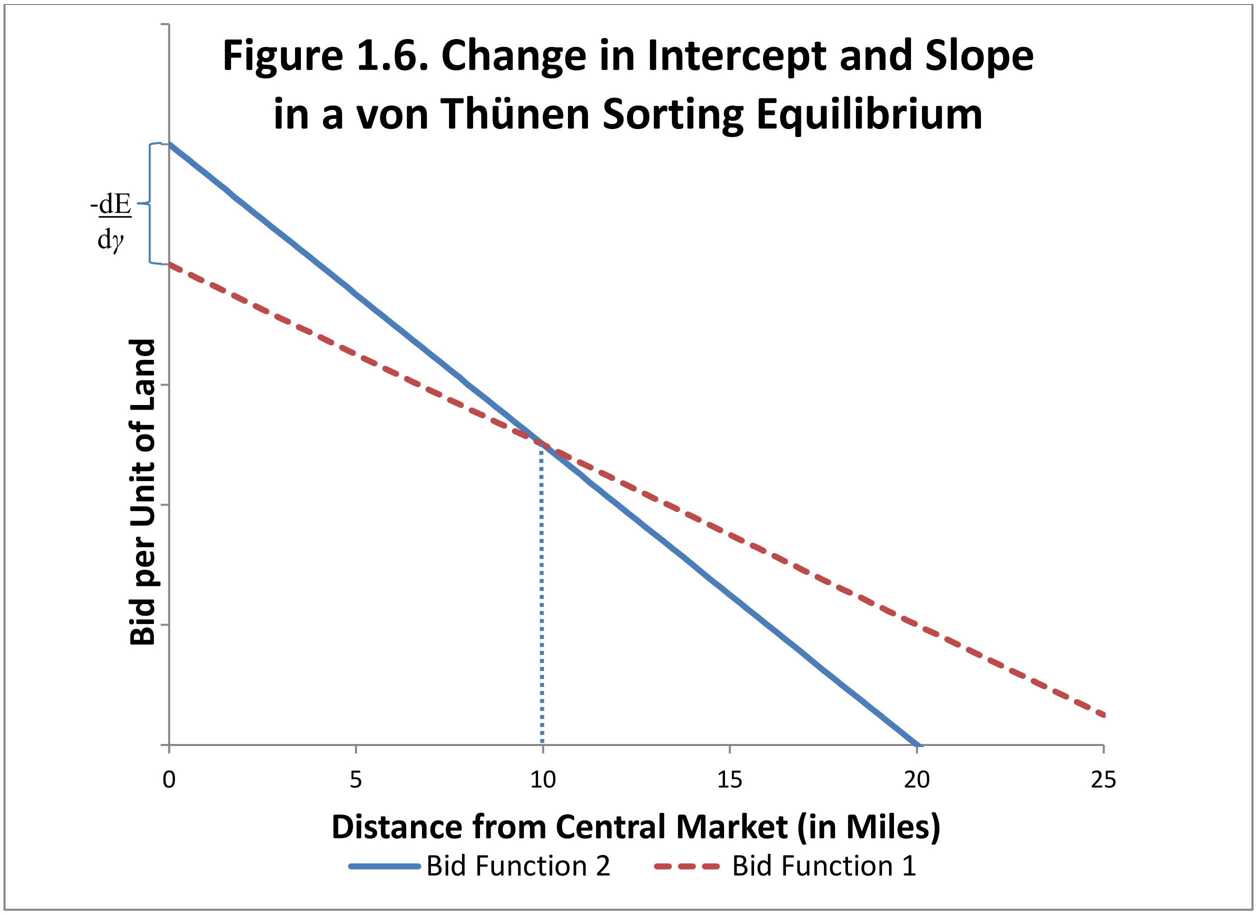


Figure 1.6. Change in Intercept and Slope
in a von Thünen Sorting Equilibrium
-dE
dγ
Bid per Unit of Land
0
5
10
15
20
25
Distance from Central Market (in Miles)
Bid Function 2
Bid Function 1

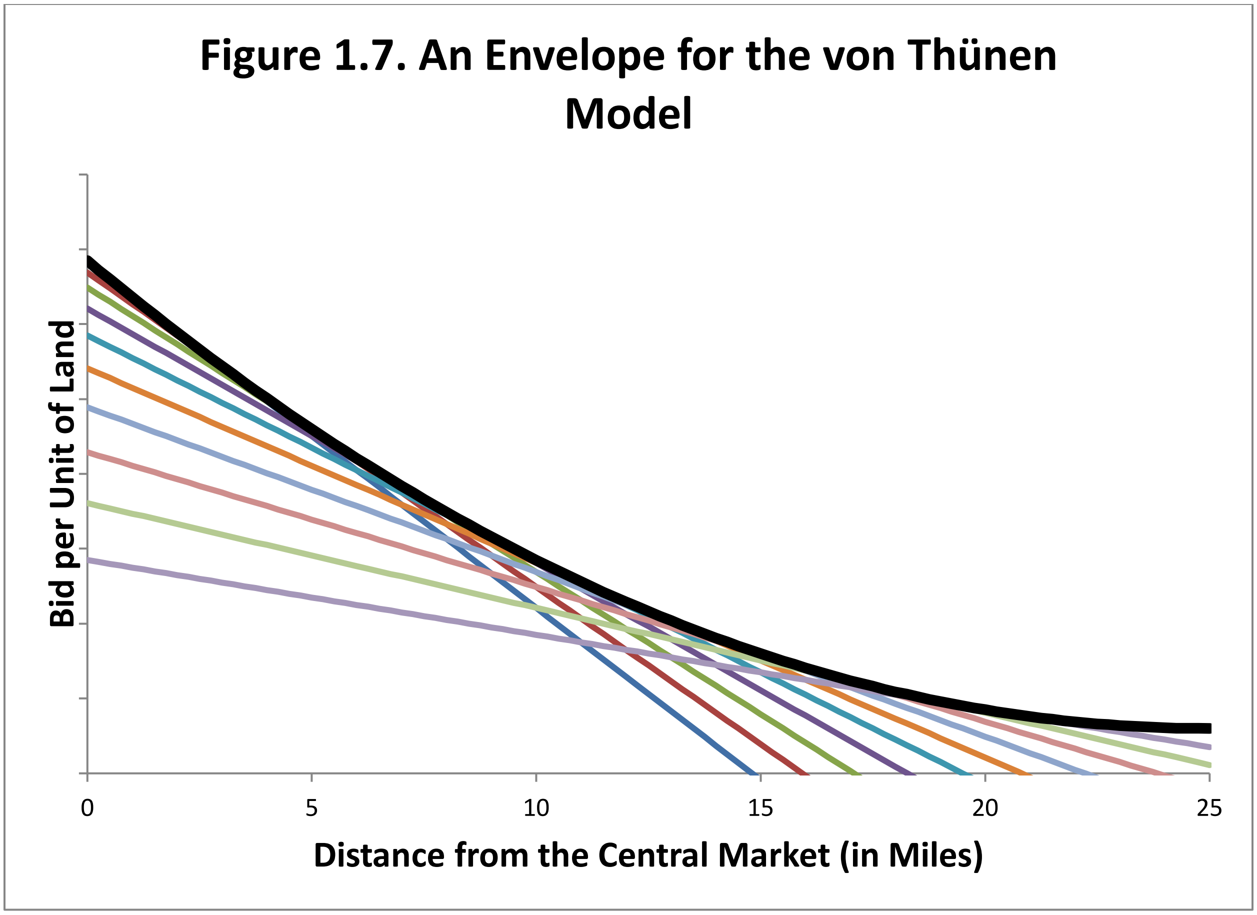


Figure 1.7. An Envelope for the von Thünen Model
Bid per Unit of Land
0
5
10
15
20
25
Distance from the Central Market (in Miles)

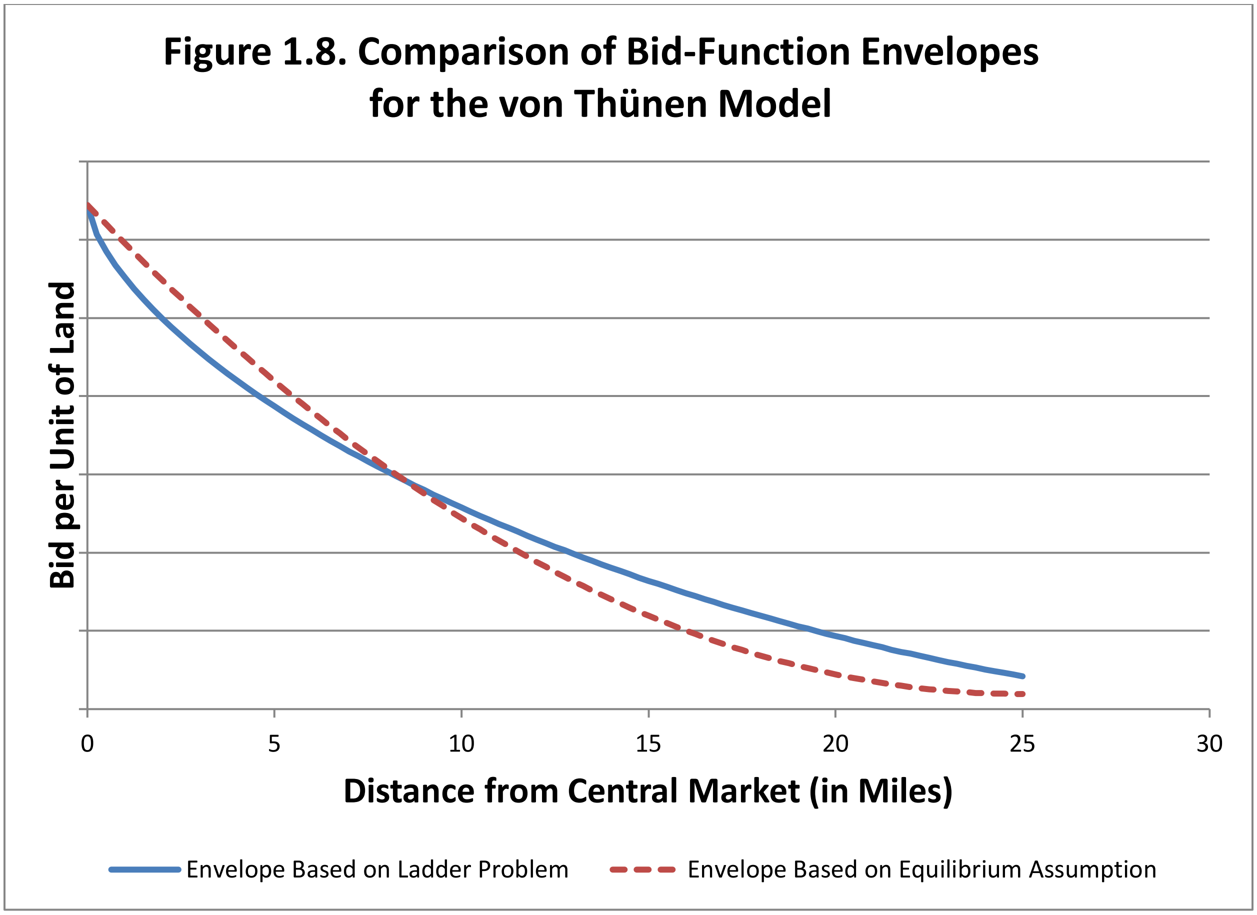


Figure 1.8. Comparison of Bid-Function Envelopes
for the von Thünen Model
Bid per Unit of Land
0
5
10
15
20
25
30
Distance from Central Market (in Miles)
Envelope Based on Ladder Problem
Envelope Based on Equilibrium Assumption

Endnotes

[1] Examples of envelopes in these fields and others can be found at the Mathematica demonstrations web site: http://demonstrations.wolfram.com .

[2] Although Alonso was trained as a regional scientist, not an economist, his 1974 book is widely regarded as one of the works that established urban economics as a sub-discipline. Moreover, the discussion of von Thünen in this chapter focuses on the views of economists. Although I am not familiar with the geography literature, it is my understanding that geographers claim von Thünen's heritage as well. See, for example, Haggett (1996).

[3] For further discussion of mathematical envelopes, see http://en.wikipedia.org/wiki/Envelope_(mathematics) or http://mathworld.wolfram.com/Envelope.html and the references they cite.

[4] This function comes from the Wikipedia citation in the previous endnote.

[5] This derivation follows the one posted by Mathematica at http://mathworld.wolfram.com/Astroid.html.

[6] This envelope is also an example of a "glissette," which is defined as "The locus of a point (or the envelope of a line) fixed in relation to a curve which slides between fixed curves. For example, if is a line segment and a point on the line segment, then describes an ellipse when slides so as to touch two orthogonal straight lines. The glissette of the line segment itself is, in this case, an astroid." http://mathworld.wolfram.com/Glissette.html.

[7] As Alonso (1964, p. 3), points out, Adam Smith and David Ricardo recognized that more productive land would command a higher rent, but neither of them recognized that the price of agricultural land might vary with distance from a market.

[8] The citation is to the third edition of von Thünen's book, which obviously appeared after his death. As indicated in the text, however, the first edition of this book, which contained the model we are interested in, appeared in 1826. See Grotewold (1959) or Samuelson (1983).

[9] This quotation and the other information in this paragraph are taken from a brief biography of von Thünen on the website of the Johann Heinrich Von Thünen-Institut. The web address is http://www.vti.bund.de/en/startseite/about-us/johann-heinrich-von-thuenen.html .

[10] This equation came from his later work on the marginal productivity of labor and had nothing to do with bid functions of envelopes. Samuelson (1983) provides a detailed discussion.

[11] Samuelson's admiration for von Thünen apparently does not rest on von Thünen's land use theory. ndeed, this is how the Samuelson article (1983, p. 1482) ends:

> Modern geographers claim Thünen. That is their right. But economists like me, who are not all that taken with location theory, hail Thünen as more than a location theorist. His theory is a theory of general equilibrium.
>
> Thünen belongs in the Pantheon with Leon Walras, John Stuart Mill, and Adam Smith. As Schumpeter would say, it is the inner ring of Valhalla they occupy.

[12] This discussion of von Thünen is based on Grotewold (1959). A similar analysis, with less detail about agricultural products, is provided by Alonso (1964, pp. 37-42).

[13] Grotewold (1959) focuses on von Thünen's bid functions and land-use predictions and does not really do justice to the full scope of von Thünen's model. As discussed by Samuelson (1983), von Thünen actually developed a general equilibrium model of city and agricultural production. For example, von Thünen assumed that residents of the city earned their income producing cloth for the farmers. The discussion in this chapter is designed simply to motivate the concept of a bid-function envelope, not to contribute to the history of economic thought. For the latter objective, the reader should turn to Samuelson (1983).

---

[14] This terminology was introduced by Alonso, who calls his version of Equation (1.15) a "bid-rent function" (1964, p. 41).

[15] This discussion leaves out several details about the outer rings. See Grotewold (1959).

[16] The outer edge of the cultivated area is, of course, analogous to the size of a metropolitan area in Alonso (1964) and the subsequent literature in urban economics.

[17] In discussing his version of the von Thünen model, for example, Alonso (1964, p. 41) says that the bid-rent "curves of all the potential users of land are compared at all locations, and the land at each location is assigned to the highest bidder. These highest bids, taken together constitute the actual rent structure." Then, on page 76, Alonso explicitly refers to the set of highest bids as an "envelope." Alonso uses similar language to discuss the way households who have bid-rent curves with different slopes, due to different incomes of some other factor, are allocated to different residential locations in a city. His Figure 30 (p. 87), for example, illustrates an equilibrium allocation and the associated envelope for three types of households. This figure provides an introduction to sorting, although Alonso does not use this term. The envelopes derived in Chapter 4 are descendants of Alonso's figure.

[18] A thoughtful review of the literature on this topic is provided by White (1999).

[19] As indicated in Note 12, my objective is to motivate the derivation of a bid-function envelope, not to build a better von Thünen model. To be honest, many of my assumptions are quite unrealistic. In contrast, Samuelson (1983) brings the tools of modern economics to bear on the von Thünen model. Samuelson not only shows that von Thünen leaves many question unanswered and but also shows how these questions can be answered formally while keeping the spirit of the original. Samuelson's conclusion is that "Conceptual difficulties in Thünen's theory of distribution between labor, land, and capital have been shown to be capable of being cleared

up while still his essential vision is preserved" (p. 1487). The urban models that are discussed in later chapters are general equilibrium models, so their debt to von Thünen goes beyond the concepts of bidding and sorting.

[20] Because all households are alike, they can all be lumped together with a collective utility, U, and a collective income, Y. With a Cobb-Douglas utility function and N commodities, the household problem is

$$\text{Maximize } U = \sum_{i=1}^{N} Q_i^{\alpha_i} \text{ subject to } Y = \sum_{i=1}^{N} P_i Q_i$$

The demand function in the text is just the quantity demanded, Q_i, that is the solution to this problem when the α_is are scaled to sum to 1.0. Demand functions are unaffected by a monotonic transformation of a utility function, so this scaling imposes no loss of generality.

[21] Land use could also be divided into wedges, with each farm operating on an arc within a given wedge.