Envelopes for Economists:
Housing Hedonics and Other Applications

An e-Book Edited by John Yinger

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Chapter 2: World-Record Envelopes

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2.0. Introduction

In his classic book on the American astronauts, Tom Wolfe (1979) reports that Navy pilots spent a lot of time talking about their flying experiences.

> One of the phrases that kept running through the conversation was "pushing the outside of the envelope." The "envelope" was a flight-test term referring to the limits of a particular aircraft's performance, how tight a turn it could make at such-and-such a speed, and so on. "Pushing the outside," probing the outer limits, of the envelope seemed to be the great challenge and satisfaction of flight test. Wolfe (1979, p. 12).

This chapter explores the efforts of elite athletes to push out their own envelopes.[1]

A mathematical envelope for a family of curves is another curve that is tangent to every member of the underlying family. Moreover, an "upper" envelope is the set of highest points for the "y" variable at each value for the "x" variable. The highest point at a given "x" is associated with a single member of the family of curves. A lower envelope is the set of lowest points for each "x." For the application of these concepts to athletics races, which are the only kind of events considered in this chapter, think of each member of a family of curves as the set of best performances for a given athlete in a series of related events of different lengths. The associated lower envelope is the set of fastest times (= lowest values with time on the vertical axis), and the upper envelope is the set of highest speeds.

The analysis in this chapter focuses exclusively on upper envelopes. Expressing world records in terms of speed, not time, makes the world-record envelopes more intuitive and easier to read. Moreover, it facilitates a focus on the speed-distance tradeoff that confronts athletes in all racing events. However, this focus does not change the substance of the analysis, which would lead to the same conclusions if lower envelopes based on time were used instead.

This chapter estimates world-record envelopes—the relationship between world record speeds and distance—for running, freestyle swimming, and ice skating. The results are used to shed light on the trade-off between speed and distance that faces elite athletes. These envelopes are compared across sports, across time, and across gender. Although estimates of this relationship have appeared in the literature, no previous study recognizes that it is estimating an envelope—and hence that individual performance curves are part of the logical structure of the problem. This chapter shows individual performance curves and discusses their mathematical form.

The analysis in this chapter is limited to racing world records, which are amenable to a relatively straightforward application of the logic of envelopes. However, the analytical techniques in this chapter might shed light on other human endeavors, as well. Automobile manufacturers face a trade-off between acceleration and fuel economy, for example, and home builders may have to trade off natural light against energy efficiency.[2] (Natural light and energy efficiency affect housing prices. I had to get housing hedonics into this chapter somehow!)

2.1. World-Record Envelopes for Racing

For the purposes of this chapter, it will prove useful to select a form for the envelope that applies to world racing records—that is, a form for the world-record envelope. A form of this type will allow us to make general statements about the speed-distance tradeoff facing elite athletes, to see how that tradeoff varies across sports, over time, and between men and women. This type of information could prove useful to athletes, coaches, scholars who study the incentives facing elite athletes, and scholars who investigate human physical limits.[3]

My analysis begins by identifying the empirical relationship between speed and distance for world records in men's track. My objective is to find the functional form that best explains this relationship as indicated by its R-squared.

The data for this exercise are presented in Table 2.1. This table lists times and speeds for 13 running events. Except for the last two, these events are run on an outdoor track. The last two are road races. Indoor events and events that are rarely staged are not included in this table.

Regressions based on these data are presented in Table 2.2. Several specifications have high explanatory power, but the winner is a regression where the explanatory variable is specified as an inverse hyperbolic cosecant of distance. As we will see, this form also works well with events other than men's track. Moreover, this form has the advantage that it is possible to identify the underlying family of individual performance curves for which it is the envelope. We return to this topic in Section 2.4.

This form can be written as

$$S^E = \alpha + \beta \ln\left\{\frac{1+\sqrt{1+D^2}}{D}\right\} \tag{2-1}$$

where S is speed (in MPH), the E superscript indicates an envelope, and D is distance (in miles). The values of α and β for a given sport determine the speed-distance tradeoff facing athletes in that sport and show how this trade-off varies a cross races of different lengths. The following section estimates these parameters for running, freestyle swimming, and speed skating.

2.2. Estimated World-Record Envelopes for Running, Swimming, and Speed Skating

World-record envelopes for running, freestyle swimming, and speed skating are presented in Table 2.3. These envelopes are estimated using the form in Equation (2.1). The coefficient of the distance term is highly significant in every case. Moreover, all of the regressions have high explanatory power, although the R-squared is lower for speed skating than for the other sports. In addition, this table reveals that distance has a much smaller impact on swimming speed, and a slightly smaller impact on skating speed, than it does on running speed.

Figure 2.1 presents the world records and the world-record envelopes for men's and women's running.[4] This figure shows that the world-record envelope provides a close approximation to world records, which is, of course, consistent with Table 2.3. Moreover, the men's and women's world-record envelopes have similar shapes. As expected, the speeds for women are somewhat smaller than those for men for every event. However, an expanded analysis (not presented) finds no significant difference in the intercepts and slopes of the men's and women's curves. The differences might well be significant with a larger sample size, but they are not very large in any case.

Further information on the speed-distance tradeoff appears in Table 2.4. This table lists running events by distance and shows the increase in speed that is required to maintain a world record performance when moving up one row in the table; that is, when moving to a shorter event.[5] A runner who matches the world record speed for 800 meters, for example, would have to increase his or her speed by about 3.0 miles per hour to match the world record for 400 meters.

The first row of Table 2.4 indicates that the world record speed at 100 meters is not much faster than the record speed for 200 meters. This result reflects the difficulty of starting, that is, of getting out of the blocks.[6] The record speeds for a 100 meter dash, 23.35 MPH for men and 21.32 MPH for women, are only slightly faster than the record speeds for a 200 meter dash, 23.31 MPH for men and 20.96 MPH for women. Moreover, the record speed for the 100 meter dash is greater than the record speeds for the 50 meter dash, 20.12 MPH for men and 18.77 for women, which is an indoor event not included in Figure 2.1. This start-up effect is difficult to see in Figure 2.1, because the triangles indicating the world records for the 100 meter and 200 meter dashes are overlapping.[7]

Table 2.4 also indicates that the speed increase required to keep up with the world record pace at a shorter distance is, not surprisingly, particularly high for races shorter than 1,000 meters. For the longer events, a runner can keep up with the world record pace at a shorter distance by maintaining a smaller speed increase for a longer period of time.

Figures 2.2 and 2.3 provide the same information for freestyle swimming and speed skating, respectively. In both of these figures, the shapes of the world-record envelopes are virtually the same for men and women, with somewhat slower speeds for women than for men. Moreover, as in the case of running, the intercepts and slopes of the men's and women's curves are not significantly different (not shown).

The start-up effect is even greater in speed skating than it is in running. In fact, the men's and women's record speeds are actually is higher over 1000 meters (34.05 MPH) than over 500 meters (33.28 MPH), and the record speed for men at 1500 meters (33.49) also exceeds their record 500 meter speed. Apparently, it takes skaters a while to get up to cruising speed. In contrast, no start-up effect is apparent for freestyle swimming.

2.3. Envelope Progressions over Time

One benefit from estimating world-record envelopes is that they can be estimated for different years. This step makes it possible to determine not only whether and by how much record speeds have increased, but also whether the speed-distance trade-off has changed. Do modern training methods or running shoes boost speed at a given distance or distance at a given speed?[8]

Figure 2.4 and Table 2.5 provide answers to these questions. The world-record envelope has clearly shifted upward over the years. This point is obvious in Figure 2.4 and is documented by the significant, growing year effects across the first row in Table 5. These results give no sign, however, that the trade-off between speed and distance has changed significantly over the last 120 years—at least in the case of men's running. The upward-shifting envelopes in Figure 2.4 appear to have the same shape, and the interaction terms for the distance variable in Table 2.5 indicate that the shape of the envelope has not changed significantly since 1900.

2.4. Individual Performance Curves and World-Record Envelopes

Figures 2.1 to 2.4 show the envelopes of underlying individual performance curves. In theory, an envelope should be tangent with a single performance curve at any given distance. In practice, however, some athletes are so gifted that they set the world record in more than one event. Nevertheless, the performance curve-envelope logic is still approximately right and provides some valuable formal structure to an analysis of athletic contests.

Figure 2.5, which applies to men's running, combines world records, the estimated world-record envelope, and the individual performance curves of several famous runners: Usain Bolt, Wilson Kipkiter, Hicham El Guerrouj, and Haile Gebrselassie. These individual performance curves are based on the personal best times of each runner. This figure also includes a performance curve for a very undistinguished runner: me. This curve provides some perspective by revealing that the fastest speeds of a mediocre college runner are 4 to 5 MPH slower than the speeds of the world record holders.[9]

This figure demonstrates that the underlying performance curves of elite runners tend to be close to linear, at least in the range of races in which they officially participate. Each of these individual performance curves also is approximately tangent to the estimated world-record envelope. This result obviously reflects the fact that these individuals are or recently were world record holders in one or more events.

To keep Figure 2.5 legible, many worthy track stars are not included. Eliud Kipchoge, the current record holder in the marathon, is left out because his profile is virtually identical to that of Haile Gebrselassie, and because Gebrselassie has better times for most events below the marathon. Among earlier stars, Jesse Owens (1913-1980) set two running world records (220 yards and 200 meters) and tied a fourth (100 yards) in less than one hour in 1935—when he was still a college student.[10] He then went on to tie the 100 meter record and break the 200 meter record at the 1936 Olympics. Paavo Nurmi of Finland (1897-1973) set world records in 22 running events for distances ranging from 1,500 to 20,000 meters.

A more formal look at individual performance curves is possible because the family of performance curves that leads to the envelope (2.1) can be identified. To be specific, this family of curves is defined by:

$$S^{I} = \alpha + \beta - \delta\beta D + \beta \ln\left\{\delta(1+\sqrt{1+D^{2}}\right\} \qquad (2\text{-}2)$$

where δ is a parameter that differs across runners and the superscript “I” indicates that the function describes an *individual* performance curve.[11]

Figure 2.6 illustrates an arbitrary set of individual performance curves based on Equation (2.2) and the associated envelope defined by Equation (2.1). These individual performance curves, like the empirical performance curves in Figure 2.5, are close to linear. Indeed, they look a lot like the performance curves in Figure 2.5 except for the fact that they are extrapolated well beyond the set of distances in which an elite runner is likely to compete. This extrapolation suggests that elite runners who specialize in events shorter than 10,000 meters would exhibit running speeds below 12 MPH if they tried to run in longer events. This sounds implausible. A more reasonable interpretation is to say that Equation (2.2) provides an accurate approximation for the racing speeds of elite athletes for the set of races in which they formally participate.

The accuracy of this interpretation can be tested in the case of elite athletes who have personal bests recorded for a wide range of track events. This type of test appears in Table 2.6. This table presents, for five elite runners, the results of a regression of the runner's personal best times in a series of running events on the event's distance using the form defined by Equation (2.2) and illustrated in Figure 2.6. For all five elite runners in this table, the explanatory power of the regression as measured by the R-squared is high and the variable defined by Equation (2.2) is highly significant. These results reinforce the conclusion that Equation (2.2) and its envelope, Equation (2.1), are highly accurate forms for approximating the relationship between speed and distance for world-record-breaking runners.

In principle, an ambitious runner might be able to use estimates of his or her personal performance curve to select training tactics and race selections. This curve could be estimated over time or with different training methods to forecast its position in the future. Moreover, it could be used to select the event on the world-record envelope (or the national-record envelope or the college conference envelope) that the athlete is most likely to reach.

2.6. Conclusions

The world-record envelope for a racing event is the set of highest speeds attained at each event distance. This chapter shows that in this envelope can be closely approximated by a regression of speed on a particular function of distance, namely, the inverse hyperbolic cotangent, which is described in Equation 2.1. The use of this form for the envelope facilitates an understanding of variation in the speed-distance tradeoff across events of different lengths and makes it possible to compare this tradeoff across sports, over time, and between men and women.

I find that the speed differences between short and long events are particularly large in running compared to swimming and speed skating. In addition, the shape of the world-record envelope is very similar for men and women in all three of these sports and very similar across time, at least for men's running.

With the functional forms used in this chapter, the speed-distance tradeoffs facing individual athletes also can be observed. The forms I derive provide highly accurate predictions of the personal best times for elite male runners regardless of the event distances in which they specialize. Extensions of this analysis could prove to be valuable training tools for runners and their coaches. How much effort, or what types of effort, would move a runner's performance curve closer to the world-record envelope or some other aspirational envelope?

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Table 2.1: World Records in Track, March 2020

Race	Distance (Miles)	Men WR Time	Men WR Speed (MPH)	Women WR Time	Women WR Speed (MPH)
100 m	0.06214	9.58s	23.3501	10.49s	21.3245
200 m	0.12427	19.19s	23.3136	21.34s	20.9648
400 m	0.24855	43.03s	20.7943	47.6s	18.7978
800 m	0.49710	1mi 40.9s	17.7359	1mi 53.3s	15.7948
1,000 m	0.62137	2mi 12.0s	16.9465	2mi 29.0s	15.0130
1,500 m	0.93206	3mi 26.0s	16.2884	3mi 50.1s	14.5824
1 mile	1.00000	3mi 43.1s	16.1363	4mi 12.3s	14.2687
2000 m	1.24275	4mi 44.8s	15.7089	5mi 23.7s	13.8211
3000 m	1.86412	7mi 20.7s	15.2277	8mi 6.1s	13.8054
5,000 m	3.10686	12mi 37.4s	14.7672	14mi 11.2s	13.1399
10,000 m	6.21373	26mi 17.5s	14.1803	18mi 51.7s	12.5459
1/2 marathon	13.10940	58mi 1.0s	13.5265	1h 6mi 11.0s	12.1290
marathon	26.21880	2h 1mi 39s	12.9316	2h 14mi 4.0s	11.7339

Source: Wikipedia (2020). “m” stands for meter; “WR” stands for world record; “h” stands for hours; “mi” stands for minutes; “s” stands for seconds; “MPH” stands for miles per hour.

Table 2.2. Alternative Forms for the Estimated WR Curve

	hy	ta	log	cs	se	quadratic	linear
Var1	3.1720	-9.6123	-1.8465	0.6498	7.3487	-0.9686	-0.2649
t-stat	20.27	-14.14	-8.86	5.87	5.66	-2.54	-2.35
Var2						0.0287	
t-stat						1.92	
Constant	13.4031	23.4476	17.2967	15.2756	13.0693	19.0867	18.1186
t-stat	55.95	45.95	49.16	26.71	15.24	19.34	19.17
R-Squared	0.9739	0.9479	0.8772	0.758	0.7445	0.5134	0.3345

Notes: Regressions for 13 events in men's track. Dependent variable is world record speed in MPH. Explanatory variable (or variables with a quadratic) are functions of distance in miles. "WR" stands for world record; "hy" stands for inverse hyperbolic cosecant; "ta" stands for hyperbolic tangent; "cs" stands for hyperbolic cosecant; "se" stands for hyperbolic secant.

Table 2.3. Estimated World-record envelopes, 2020

	Track		Swimming		Skating	
	Men	Women	Men	Women	Men	Women
Intercept	13.403	11.903	3.276	3.202	30.115	27.511
t-stat	55.949	21.415	19.420	22.854	38.678	41.064
hy	3.172	2.935	0.452	0.330	2.288	2.264
t-stat	20.271	21.415	7.293	6.420	3.032	3.487
R-squared	0.974	0.977	0.930	0.912	0.697	0.752
Observations	13	13	6	6	6	6

Notes: Regressions with speed (MPH) as the dependent variable and the hyperbolic arc cosecant ("hy") of distance (in miles) as the explanatory variable.

Table 2.4. The Speed-Distance Trade-Off in World Records for Track

Comparison Event	Event	Cut in Distance (in Miles)	Required Increase in Speed (MPH) Men	Women
200 m	100 m	0.06214	0.0365	0.3597
400 m	200 m	0.12427	2.5194	2.1669
800 m	400 m	0.24855	3.0583	3.0030
1,000 m	800 m	0.12427	0.7894	0.7818
1,500 m	1,000 m	0.31069	0.6581	0.4306
1 mile	1,500 m	0.06794	0.1521	0.3137
2000 m	1 mile	0.24275	0.4274	0.4476
3000 m	2000 m	0.62137	0.4812	0.0156
5,000 m	3000 m	1.24275	0.4604	0.6655
10,000 m	5,000 m	3.10686	0.5869	0.5940
1/2 marathon	10,000 m	6.89567	0.6538	0.4170
marathon	1/2 marathon	13.10940	0.5949	0.3951

Notes: This table shows the increase in speed that is needed to stay on the world-record envelope when moving from the comparison even to the indicated event. These results are based on Table 2.1.

Table 2.5. Time Progression in the World-Record Envelope in Men's Track

		Interactions with Year		
		1940	1980	2020
Intercept	10.6261	1.1494	2.3412	2.8415
t-stat	23.6862	1.8117	3.6902	4.4788
hy	3.2716	-0.0190	-0.1439	-0.1105
t-stat	12.8820	-0.0530	-0.4007	-0.3077
R-squared	0.9608			
Observations	36			

Notes: This table regresses men's world-record speed in 13 events on the hy term defined by equation (2.1), with year dummies and interactions between those dummies and hy.

Table 2.6. Estimates of Individual Performance Curves, Men's Track

	Bolt	Kipkiter	El Guerrouj	Gebrselassie	Kipchoge
Constant	25.1875	20.4250	16.9227	16.5014	16.3643
t-stat	43.8428	-23.4569	-13.6499	-20.8920	-9.7947
Curve	-20.2377	-5.6024	-0.8231	-1.1612	-1.1257
t-stat	-5.5591	-23.4569	-13.6499	-20.8920	-9.7947
R-Squared	0.9115	0.9964	0.9739	0.9887	0.9411
Observations	5	4	7	8	7

Note: This table presents regressions of an elite athlete's personal best times in several events on the distance variable defined by Equation (2.2).

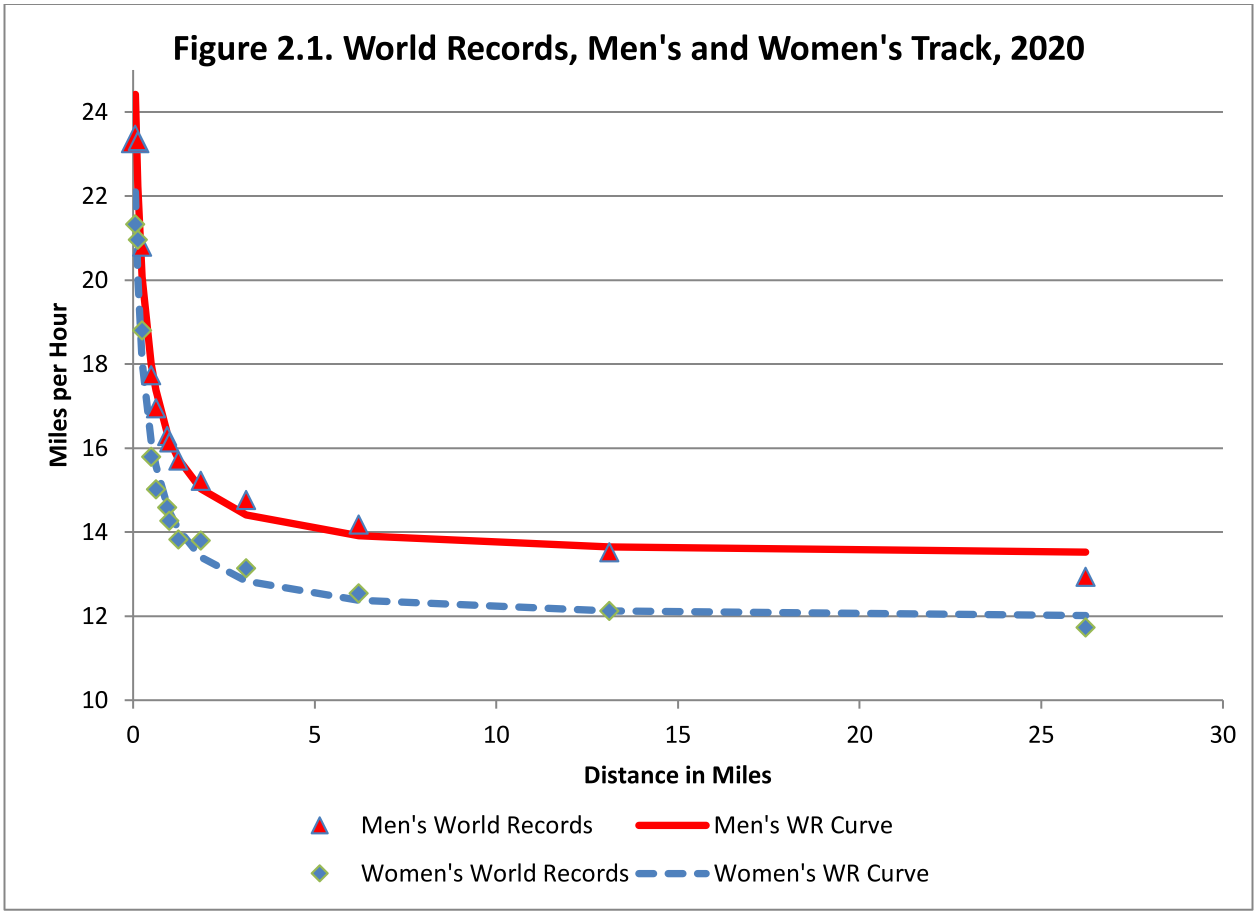
Figure 2.1. World Records, Men's and Women's Track, 2020
Miles per Hour
24
22
20
18
16
14
12
10
0
5
10
15
20
25
30
Distance in Miles
Men's World Records
Men's WR Curve
Women's World Records
Women's WR Curve

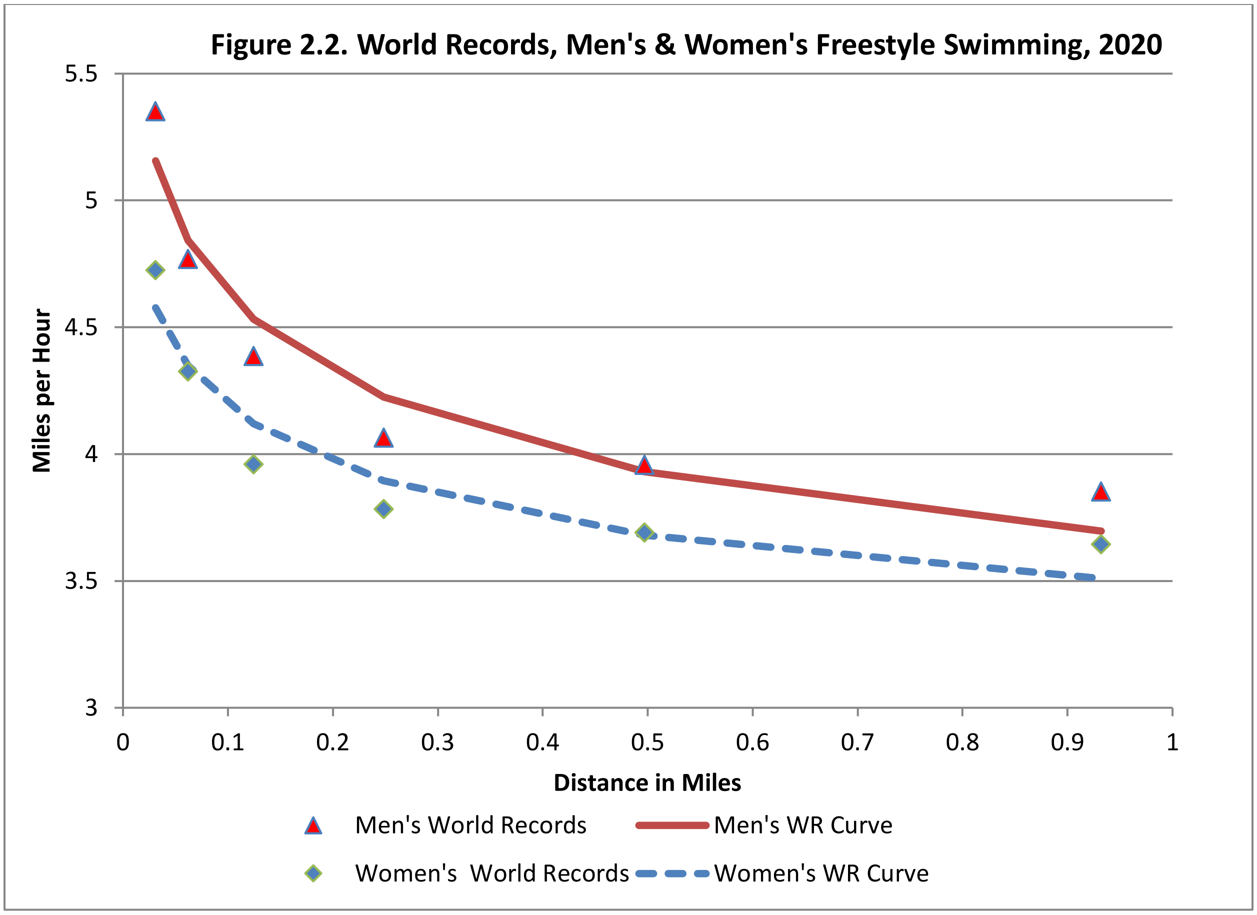
Figure 2.2. World Records, Men's & Women's Freestyle Swimming, 2020
Miles per Hour
5.5
5
4.5
4
3.5
3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Distance in Miles
Men's World Records
Men's WR Curve
Women's World Records
Women's WR Curve

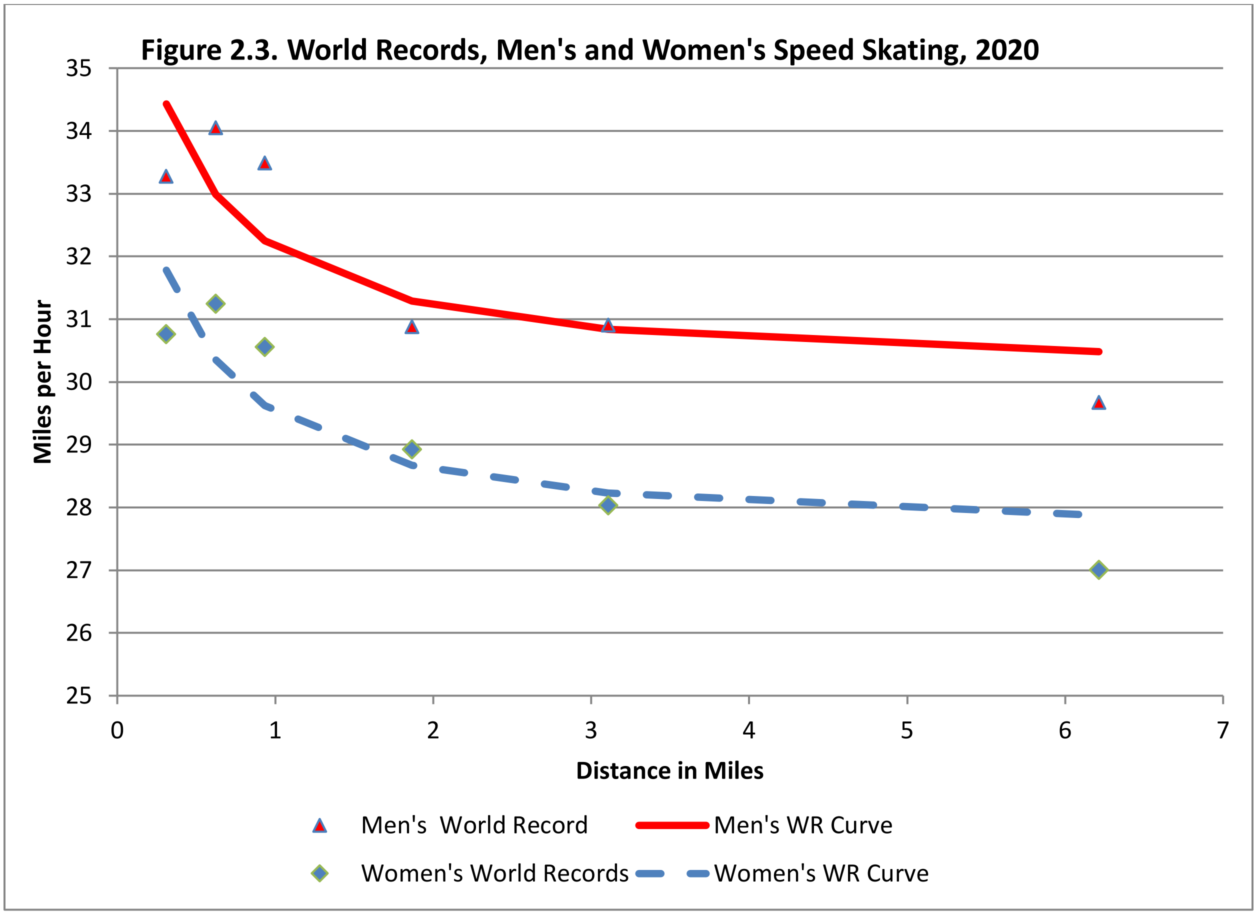
Figure 2.3. World Records, Men's and Women's Speed Skating, 2020
Miles per Hour
35
34
33
32
31
30
29
28
27
26
25
0
1
2
3
4
5
6
7
Distance in Miles
Men's World Record
Men's WR Curve
Women's World Records
Women's WR Curve

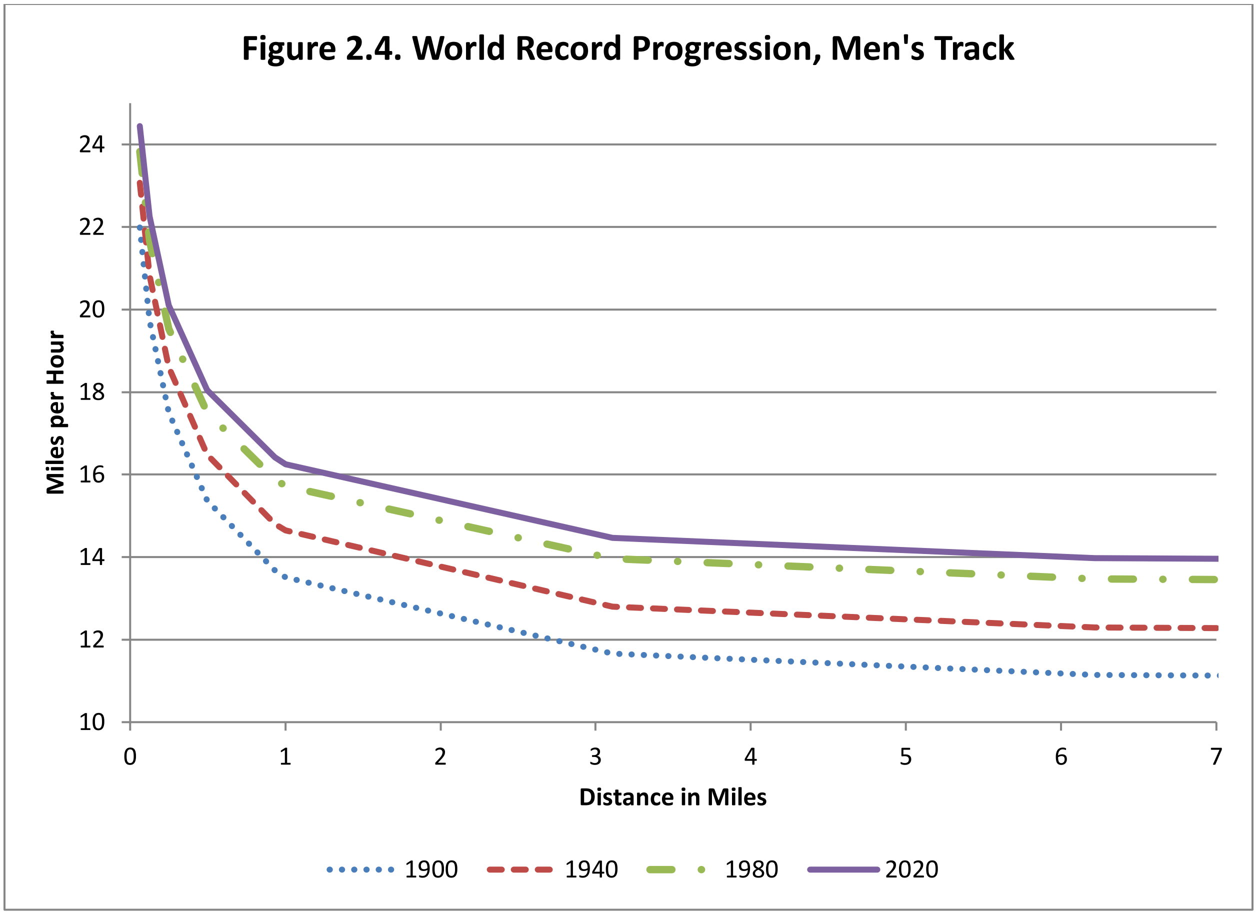
Figure 2.4. World Record Progression, Men's Track
Miles per Hour
24
22
20
18
16
14
12
10
0
1
2
3
4
5
6
7
Distance in Miles
1900
1940
1980
2020

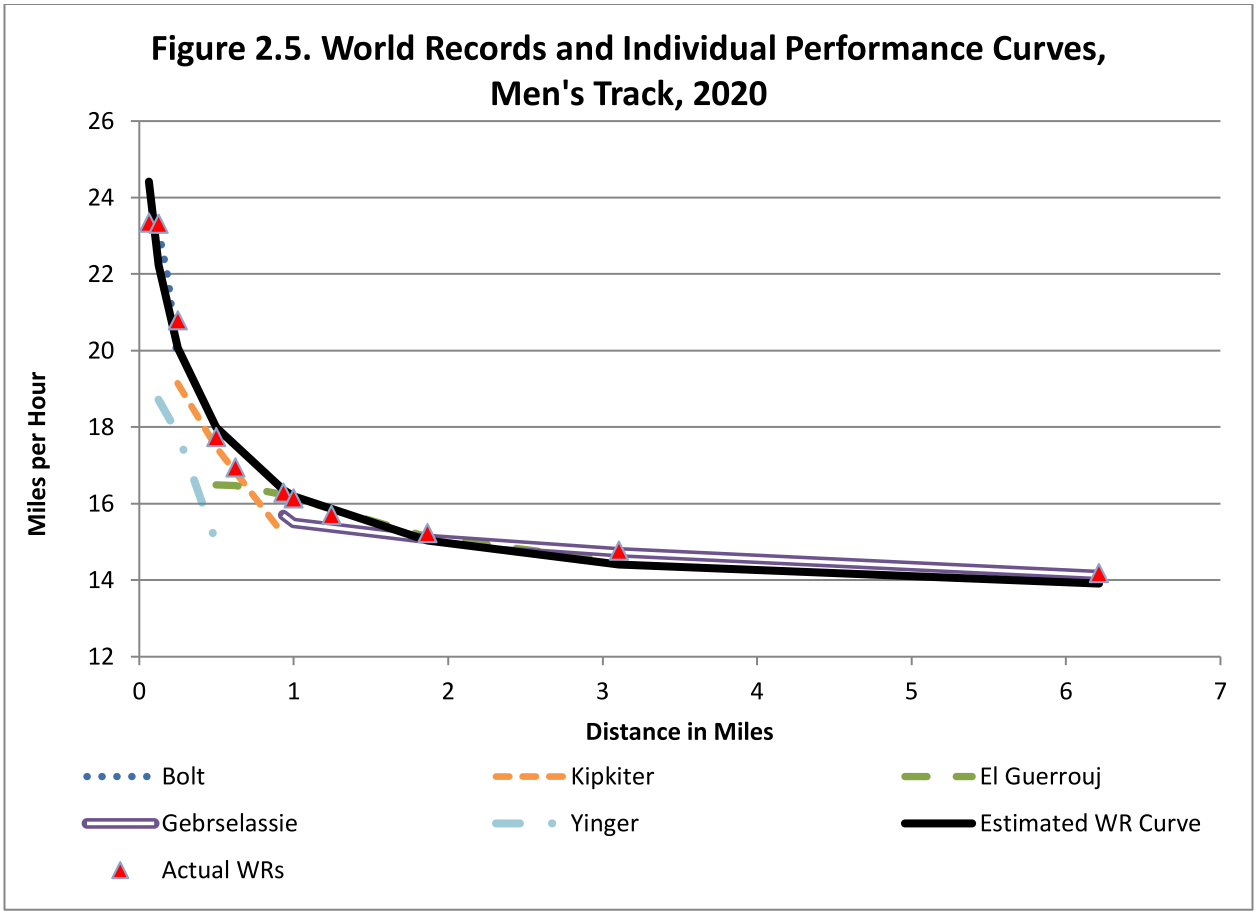

Figure 2.5. World Records and Individual Performance Curves, Men's Track, 2020
Miles per Hour
26
24
22
20
18
16
14
12
0
1
2
3
4
5
6
7
Distance in Miles
Bolt
Kipkiter
El Guerrouj
Gebrselassie
Yinger
Estimated WR Curve
Actual WRs

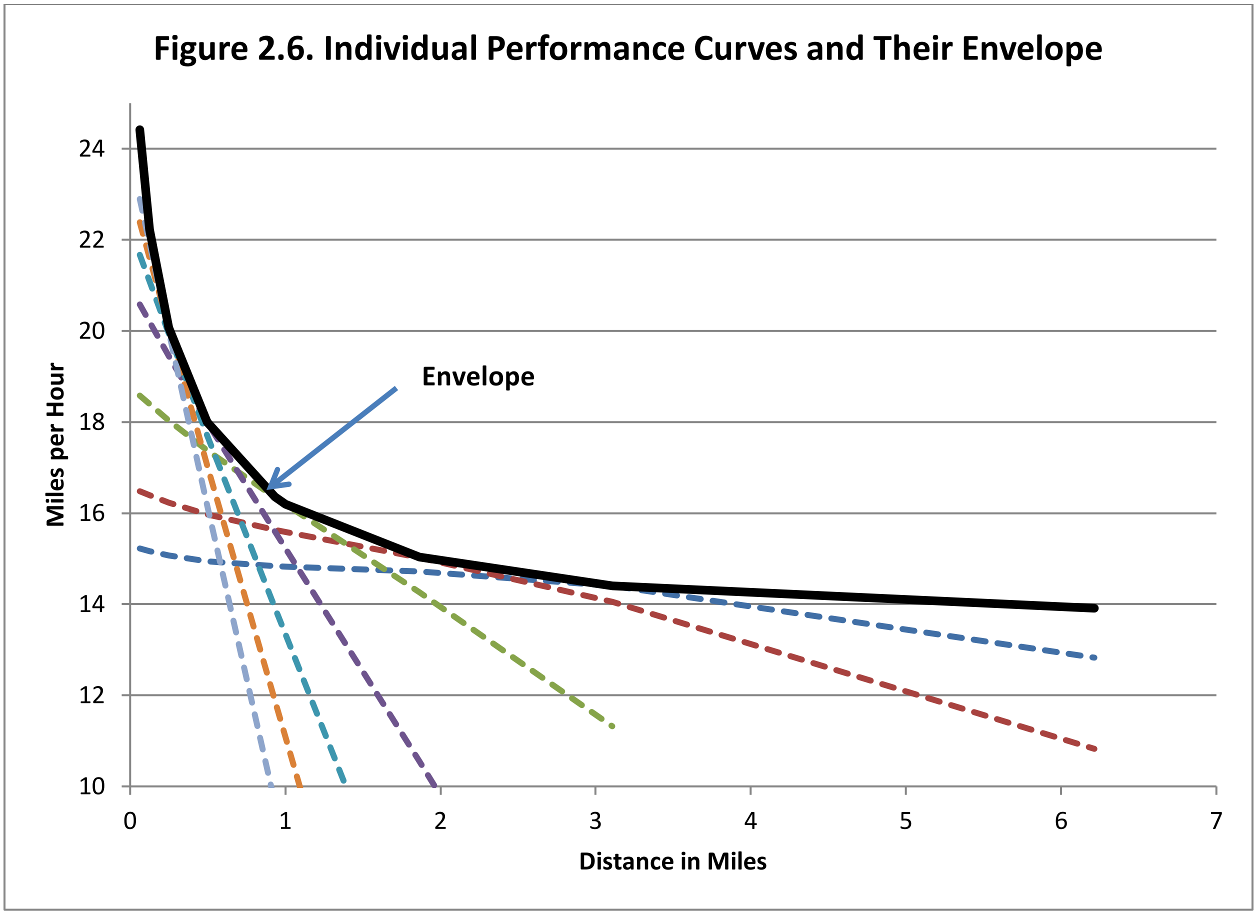
Figure 2.6. Individual Performance Curves and Their Envelope
Envelope
Miles per Hour
24
22
20
18
16
14
12
10
0
1
2
3
4
5
6
7
Distance in Miles

Endnotes

[1] Of course the envelopes in Wolfe's book, unlike the ones in this chapter, concern matters of life and death. As Wolfe puts it (1969, p. 12) "At first 'pushing the outside of the envelope' was not a particularly terrifying phrase to hear. It sounded once more as if the boys were just talking about sports." Wolfe then goes on to describe airplane crashes that took the lives of many pilots.

[2] See, for example, Cheah et al. (2009).

[3] This chapter is by no means the first attempt to examine the relationship between world record times or speed and distance. See, for example, Vandewalle (2017). So far as I know, however, no previous study has investigated marginal effects or the logic of envelopes.

[4] Please note that the ranges of the axes in this figure—and the following ones—are restricted to make the figure easier to read. Note also that the data for the regressions are not truncated; that is, the regressions use all the data in Table 1.

[5] An alternative method for determining how the speed-distance tradeoff changes as the event distance changes is by looking at the derivative of the world record envelope with respect to distance. This approach yields dS/dD at any distance. The trouble with this approach is that it applies to small changes in D, which do not correspond to differences in event lengths. Thus, these derivatives are difficult to interpret.

[6] With one exception, the results in this table can be approximated using speeds predicted by the world-record envelope. The exception is in the first row, where the required speeds are much higher for the estimated envelope, which does not formally account for the start-up problem.

[7] Another example of the start-up effect comes from Usain Bolt, who ran a 150 meter race in 2009. He was timed in 8.7 seconds for the last 100 meters of this race, which indicates a running speed of 25.71 MPH. Wow!

[8] Many previous studies have examined the progression of world records over time, often to project the likely future path of the record in individual events. See, for example Nevill and Whyte (2005), Nevill, Whyte, and Peyrebrune (2007), Berthelot et al. (2008), Kuper and Sterken (2003). This chapter makes no attempt to predict the future. So far as I know, no previous study has examined the time path of the world-record envelope.

[9] I ran track at Swarthmore College, a small liberal arts college. My best performance was breaking 2 minutes in the half mile—in my final race.

[10] What's more, within the same 45 minutes Owens set two world records in the low hurdles (220 yards and 200 meters) and another in the long jump.

[11] An envelope for a family of curves $y=\mathrm{f}\{x, \delta\}$ is the equation that satisfies $y-\mathrm{f}\{x, \delta\} = 0$ and $\mathrm{df}/\mathrm{d}\delta = 0$. Applying this rule to equation (2.2) leads to equation (2.1).